Fall 05 Name: ______________________________

Math 4581 Test 1 Bonetto

1) The motion of a pendulum is described by the following equation:

$$\ddot{x}(t) + 8t\dot{x}(t) + 4(5 + 4t^2)x(t) = \exp(-2t^2)\sin(t)$$

a) find the general solution for the equation. (**Hint**: try the substitution $x(t) = \exp(-2t^2)y(t)$.)
*Using the substitution one has:*

$$\dot{x}(t) = -4t\exp(-2t^2)y(t) + \exp(-2t^2)\dot{y}(t)$$
$$\ddot{x}(t) = (16t^2 - 4)\exp(-2t^2)y(t) - 8t\exp(-2t^2)\dot{y}(t) + \exp(-2t^2)\ddot{y}(t)$$

*Substituting into the equation we get*

$$\exp(-2t^2)\left(\ddot{y}(t) + 4y(t)\right) = \exp(-2t^2)\sin(t)$$

*or*

$$\ddot{y}(t) + 16y(t) = \sin(t)$$

*whose general solution is*

$$y(t) = A_1\cos(4t) + A_2\sin(4t) + \frac{1}{15}\sin(t)$$

*so that the general solution is*

$$y(t) = \exp(-2t^2)\left(A_1\cos(4t) + A_2\sin(4t) + \frac{1}{15}\sin(t)\right).$$

b) You want to solve the equation with boundary conditions

$$x(0) = 0 \qquad \dot{x}(\pi) = 0.$$

Find the solution.
$x(0) = 0$ *implies* $A_1 = 0$. $\dot{x}(\pi) = \exp(-2\pi^2)\left(4A_2 - \frac{1}{15}\right) = 0$ *which implies* $A_2 = \frac{1}{60}$

c) **(Bonus)** Write the Green function for the boundary conditions of point b). (**Hint**: You wrote an equation for $y(t)$. Compute the Green function for $y(t)$ and ...)

*Has we saw the equation*

$$\ddot{x}(t) + 8t\dot{x}(t) + 4(5 + 4t^2)x(t) = f(t)$$

*can be transformed in*

$$\ddot{y}(t) + 16y(t) = \exp(2t^2)f(t)$$

*The boundary condition become*

$$y(0) = 0 \qquad -4\pi y(\pi) + \dot{y}(\pi) = 0$$

*We can find the two solution satisfying one boundary condition each fixing*

$$y_1(t) = \sin(4t) \qquad y_2(t) = \cos(4t) + \pi \sin(4t)$$

*The Wronskian becomes* $W(t) = -4$ *so that we have*

$$\widetilde{G}(t,s) = -\frac{1}{4}\begin{cases} \sin(4s)\big(\cos(4t) + \pi\sin(4t)\big) & 0 < s < t \\ \sin(4t)\big(\cos(4s) + \pi\sin(4s)\big) & t < s < \pi \end{cases}$$

*From the sobstitution we get that*

$$G(t,s) = \exp(-2t^2)\widetilde{G}(t,s)\exp(2s^2)$$

*is the Green function for* $x(t)$.

2) Compute the Foureir series of the function:

$$f(x) = \begin{cases} -x & -1 \le x \le 0 \\ \\ 3x & 0 \le x \le 1 \end{cases}$$

You may use that for $-1 \le x \le 1$ we have:

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \sin(n\pi x)$$

$$|x| = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n^2\pi^2} \cos(n\pi x)$$

*Observe that*

$$f(x) = 2|x| + x$$

*so that*

$$f(x) = 1 + \sum_{n=1}^{\infty} \left( \frac{4((-1)^n - 1)}{n^2\pi^2} \cos(n\pi x) + \frac{2(-1)^{n+1}}{\pi n} \sin(n\pi x) \right)$$

3) Let $f_e(x)$ be the even extension of

$$f(x) = \sin\left(\frac{x}{2}\right) \qquad 0 \le x \le \pi$$

a) Find the Fourier series for $f_e(x)$. Does it converge pointwise? Uniformly? (Remeber that:

$$\int \sin(\lambda x)\cos(\mu x)dx = \frac{\cos(\mu-\lambda)x}{2(\mu-\lambda)} - \frac{\cos(\mu+\lambda)x}{2(\mu+\lambda)}$$

if $\mu \ne \lambda$)

*The function is clearly even so we just have to compute*

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin\left(\frac{x}{2}\right)dx = \frac{2}{\pi}$$

*and*

$$a_n = \frac{2}{\pi}\int_0^\pi \sin\left(\frac{x}{2}\right)\cos(nx)dx = -\frac{1}{\pi}\left(\frac{1}{n-\frac{1}{2}} - \frac{1}{n+\frac{1}{2}}\right) = -\frac{4}{\pi}\frac{1}{4n^2-1}$$

*so that*

$$f_e(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{4n^2-1}\cos(nx)$$

*The function is continuous and has sectionally continuous derivative so that this Fourier series converge uniformly. Obesrve moreover that* $a_n \simeq \frac{1}{n^2}$.

b) Find the Fourier series for $f_e'(x)$. Does it converge pointwise? Uniformly?

*We immediately find that*

$$f_e'(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{n}{4n^2-1}\sin(nx)$$

*The function is sectionally continuous and has sectionally continuous derivative so that this Fourier series converge pointwise. The function is not continuous so that there cannot be uniform convergence. Obesrve moreover that* $a_n \simeq \frac{1}{n}$.

c) Write an expression for $f_e''(x)$ and its Fourier series. (**Hint**: remeber the discontinuity at $x = \dfrac{\pi}{2}$)
*We can write*

$$f_e''(x) = -\frac{1}{4}\left|\sin\left(\frac{x}{2}\right)\right| + \delta(x)$$

*and*

$$f_e''(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{n^2}{4n^2-1}\cos(nx)$$

d) **(Bonus)** Can you use the relation between $f_e(x)$ and $f_e''(x)$ to compute the Foureir series of $f_e(x)$?
*We have that*

$$f_e''(x) = -\frac{1}{4}f_e(x) + \delta(x)$$

*Moreover both side are even so only $a_n$ are present. The above equation, in Fourier series, reads*

$$0 = -\frac{1}{4}a_0 + \frac{1}{2\pi} \qquad - n^2 a_n = -\frac{1}{4}a_n + \frac{1}{\pi}$$

*The above equation immediatly imply point a).*