1. Consider the periodic function

$$f(x) = x(|x| - 1) \qquad -1 \leq x \leq 1$$

(i) Does $f(x)$ has any symmetry?
We clearly have $f(-x) = -f(x)$ so that f is odd.

(ii) Is it continuous? Is it sectionally continuous and sectionally smooth?
f is continuous for $-1 < x < 1$. Moreover $f(-1) = f(1) = 0$ so that f is continuous and thus sectionally continuous. $f'(x)$ exists and is continuous for every x. Thus f is sectionally smooth.

(iii) Compute $f'(x)$ and $f''(x)$. Are them continuous, sectionally continuous, sectionally smooth?

$$f'(x) = 2|x| - 1 \qquad f''(x) = \begin{cases} -2 & -1 < x < 0 \\ 2 & 0 < x < 1 \end{cases}$$

so that f' is continuous and sectionally smooth while f'' is only sectionally smooth.

(iv) Compute the Fourier series of $f(x)$, $f'(x)$ and $f''(x)$. *We have*

$$f''(x) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n\pi} \sin(n\pi x)$$

from which we get

$$f'(x) = -\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2} \cos(n\pi x)$$

since we clearly have $\int_{-1}^{1} f'(x)dx = 0$. *Finally we have*

$$f(x) = -\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3\pi^3} \sin(n\pi x)$$

(v) What can you say on the convergence of the Fourier series for $f(x)$, $f'(x)$ and $f''(x)$?
Clearly the F.S. for f anf f' converge uniformly while the F.S. for f'' converges only pointwise.

(vi) Let

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin \frac{2n\pi}{a} x + \sum_{n=1}^{\infty} b_n \cos \frac{2n\pi}{a} x$$

Compute:

$$\sum_{n=1}^{\infty} n^2(a_n^2+b_n^2) \qquad \sum_{n=1}^{\infty} n^4(a_n^2+b_n^2)$$

From Parceval equality we get:

$$\sum_{n=1}^{\infty} n^2(a_n^2+b_n^2) = \frac{1}{\pi^2}\int_{-1}^{1} f'(x)^2 dx = \frac{2}{\pi^2}\int_0^1 (2x-1)^2 dx = \frac{2}{3\pi^2}$$

and

$$\sum_{n=1}^{\infty} n^4(a_n^2+b_n^2) = \frac{1}{\pi^4}\int_{-1}^{1} f''(x)^2 dx = \frac{2}{\pi^4}\int_0^1 4dx = \frac{8}{\pi^4}$$

(vii) (Bonus) Let

$$f_N(x) = a_0 + \sum_{n=1}^{N} a_n \sin\frac{2n\pi}{a}x + \sum_{n=1}^{N} b_n \cos\frac{2n\pi}{a}x$$

Give an estimate of

$$\sup_x |f(x)-f_N(x)|$$

and

$$\int_{-1}^{1} |f(x)-f_N(x)|^2 dx$$

2. Let $f(x)$ be a continuous function of period a with Fourier series given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin\frac{2n\pi}{a}x + \sum_{n=1}^{\infty} b_n \cos\frac{2n\pi}{a}x$$

(i) Find the Fourier series of

$$g(x) = \frac{f(x)+f(-x)}{2}$$

$$g(x) = a_0 + \sum_{n=1}^{\infty} b_n \cos\frac{2n\pi}{a}x$$

(ii) Find the Fourier series of

$$g(x) = \frac{f(x)-f(-x)}{2}$$

$$g(x) = \sum_{n=1}^{\infty} a_n \sin \frac{2n\pi}{a} x$$

(iii) Find the Fourier series of

$$g(x) = f\left(2x + \frac{a}{2}\right)$$

$$f\left(2x + \frac{a}{2}\right) = a_0 + \sum_{n=1}^{\infty} a_n \sin\left(\frac{4n\pi}{a}x + n\pi\right) + \sum_{n=1}^{\infty} b_n \cos\left(\frac{4n\pi}{a}x + n\pi\right)$$

Observe that $\cos(x + n\pi) = (-1)^n \cos x$ *and* $\sin(x + n\pi) = (-1)^n \sin x$ *so that we can write*

$$f\left(2x + \frac{a}{2}\right) = a_0 + \sum_{n=1}^{\infty} c_n \sin \frac{2n\pi}{a} x + \sum_{n=1}^{\infty} d_n \cos \frac{2n\pi}{a} x$$

where

$$c_n = \begin{cases} (-1)^{\frac{n}{2}} a_n & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad d_n = \begin{cases} (-1)^{\frac{n}{2}} b_n & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

3. The oscillation $u(t)$ of a pendulum are desribed by the equation

$$\ddot{u}(t) + \omega^2 u(t) = \cos(t)$$

Suppose the pendulum is initially at rest at its minimum, i.e. $u(0) = 0$. You want to hit it at time 0 in such a way that after 1 second the pendulum will be back at the minimum position, i.e. $u(1) = 0$. Which velocity $\dot{u}(0)$ should you give to the pendulum at time 0?
We first find a solution of the non homogenous equation. An easy guess is $u_p(t) = a\cos(t)$. *Substituting into the equation we get* $a = 1/(\omega^2 - 1)$. *Two indipendent solution of the homogenous equation are* $u_1(t) = \cos(\omega t)$ *and* $u_2(t) = \sin(\omega t)$ *so that the general solution is:*

$$u(t) = a_1 \cos(\omega t) + a_2 \sin(\omega t) + \frac{\cos t}{\omega^2 - 1}$$

The first boundary condition implies $a_1 = -1/(\omega^2 - 1)$ *while the second one gives:*

$$a_2 = \frac{\cos\omega - \cos 1}{(\omega^2 - 1)\sin\omega}$$

from which

$$\dot{u}(0) = \frac{\omega(\cos\omega - \cos 1)}{(\omega^2 - 1)\sin\omega}$$