1.11. The Poincaré Bendixson Theorem.

For an autonomous differential equation in the plane, the ω-limit set and α-limit set of bounded orbits have a very simple structure as was observed by Poincaré and Bendixson at the turn of the century. In this section, we state these results precisely leaving the details of the proofs to the reader since they are rather simple and serve as instructive exercises We give an application to the existence of a periodic orbit of van der Pol's equation and discuss in some detail the Esaki diode.

The *Jordan Curve Theorem* will play a crucial role. It is one of those geometrically obvious results whose proof is very difficult. Recall that a Jordan curve is the homeomorphic image of the unit circle in the plane.

Theorem 11.1. *(Jordan Curve Theorem) A Jordan curve in $\mathbb{R}^2$ separates $\mathbb{R}^2$ into two connected components, one bounded which is called the interior, and the other unbounded which is called the exterior.*

We consider throughout this section the differential equation (6.1) in $\mathbb{R}^2$. If ξ is a regular point of f, let Σ be a transversal to f at ξ. The transversal Σ can be considered as the diffeomorphic image of a line segment and therefore we can order the points on Σ with the natural order of the real numbers. We will assume always that we have made this ordering.

Lemma 11.1. *If $f \in C^r(\mathbb{R}^2, \mathbb{R}^2)$, $r \geq 1$, ξ is a regular point of f, Σ is transverse to f at ξ and π is the corresponding Poincaré map, then π is monotone on $D(\pi)$.*

Proof. If $D(\pi) = \emptyset$, then there is nothing to prove. If $D(\pi) \neq \emptyset$ and $\eta \in D(\pi)$, let $\pi\eta = \varphi^{\tau(\eta)}(\eta)$. Let $\eta_1, \eta_2 \in D(\pi)$ with $\eta_1 \leq \eta_2$ and suppose that $\pi\eta_1 \geq \eta_1$. We can define a Jordan Curve to be the set $\mathcal{C} = \{\, x = \varphi^t(\eta_1) : 0 \leq t \leq \tau(\eta_1) \,\} \cup [\eta_1, \pi\eta_1]$. From the Jordan curve theorem and the uniqueness of the solutions of the initial value problem for (6.1), it is now easy to show that $\pi\eta_2 \geq \pi\eta_1$. The other case where $\pi\eta_1 \leq \eta_1$ is treated in exactly the same way.

Lemma 11.2. *(Flow Box Theorem) In a sufficiently small neighborhood of a regular point of the planar system (6.1), there is a differentiable change of coordinates $y = y(x)$ such that the original system becomes the product system $\dot{y}_1 = 1$, $\dot{y}_2 = 0$.*

Proof. To simplify the notation in this proof, we will write an element in $\mathbb{R}^2$ as $x = (x_1, x_2)$. Without loss of generality, we may assume that the regular point under consideration is the origin 0 and that $f(0) = (1, 0)$. If we define $h(y_1, y_2) = \varphi^{y_1}(0, y_2)$, then $h(0, y_2) = (0, y_2)$ and $\partial h(0,0)/\partial y_1 = f(0) = (0, 1)$. Therefore, the Implicit Function Theorem implies that h maps diffeomorphically an open neighborhood U of $(0,0)$ in the (y_1, y_2)-plane to an open neighborhood V of $(0,0)$ in the (x_1, x_2)-plane. For $(x_1^0, x_2^0) \in V$, let $h(y_1^0, y_2^0) = (x_1^0, x_2^0)$. Then $h^{-1}\varphi^t(x_1^0, x_2^0) = h^{-1}\varphi^{t+y_1^0}(0, y_2^0) = h^{-1}h(t + y_1^0, y_2^0) = (t + y_1^0, y_2^0)$. This shows that the flow in the (y_1, y_2) coordinate system is the one stated in the lemma.

Lemma 11.3. *If ξ is a regular point of the planar vector field f and Σ is a transversal to f at ξ, then $\omega(\xi)$ can intersect Σ in at most one point. Moreover, if $\omega(\xi) \cap \Sigma = \xi_0$, then either $\omega(\xi_0) = \alpha(\xi_0)$ is a periodic orbit or there is a sequence $\{ t_k \}$, $t_k \to \infty$ as $k \to \infty$ such that $\varphi^{t_k}(\xi) \in \Sigma$ and $\varphi^{t_k}(\xi) \to \xi_0$ monotonically as $k \to \infty$.*

Proof. There is a sequence $\{ t'_k \}$, $t'_k \to \infty$ as $k \to \infty$ such that $\varphi^{t'_k}(\xi) \to \xi_0$ as $k \to \infty$. Lemma 11.2 implies that the orbit through $\varphi^{t'_k}(\xi)$ must intersect Σ at some point, say $\varphi^{t_k}(\xi)$ and $\varphi^{t_k}(\xi) \to \xi_0$ as $k \to \infty$. If two distinct points in this sequence coincide, then they all coincide and the orbit through ξ is periodic. If all of the points in this sequence are distinct, then we can reorder so that $t_k < t_{k+1}$ and $\pi(\varphi^{t_k}(\xi)) = \varphi^{t_{k+1}}(\xi)$. From Lemma 11.1, it follows that $\varphi^{t_k}(\xi) \to \xi_0$ monotonically which shows also that $\omega(\xi)$ can intersect Σ in at most one point.

Lemma 11.4. *If ξ_0 is a regular point of the planar vector field f and $\xi_0 \in \omega(\xi) \cap \gamma^+(\xi)$, then $\gamma^+(\xi)$ is a periodic orbit.*

Proof. If $\xi_0 \in \omega(\xi) \cap \gamma^+(\xi)$ and Σ is a transversal to f at ξ_0 and $\omega(\xi) \neq \gamma^+(\xi)$, then Lemma 11.3 implies that there is a sequence $\{ t_k \}$, $t_k \to \infty$ as $k \to \infty$ such that $\varphi^{t_k}(\xi) \in \Sigma$ and $\varphi^{t_k}(\xi) \to \xi_0$ monotonically as $k \to \infty$. This is clearly a contradiction since $\xi_0 \in \gamma^+(\xi)$ and $\omega(\xi)$ can intersect Σ in at most one point.

Lemma 11.5. *If M is a bounded nonempty minimal set of the planar system (6.1), then M is either a critical point or a periodic orbit.*

Proof. If γ is an orbit in M, then $\alpha(\gamma)$ and $\omega(\gamma)$ are nonempty and belong to M. Since $\alpha(\gamma)$ and $\omega(\gamma)$ are invariant, we have $\alpha(\gamma) = \omega(\gamma) = M$. If M contains a critical point ξ, then $M = \xi$. If $M = \omega(\gamma)$ does not contain a critical point, then $\gamma \subset \omega(\gamma)$ implies that γ and $\omega(\gamma)$ have a point in common and Lemma 11.4 implies that γ is periodic.

Lemma 11.6. *If $\omega(\xi)$ contains regular points and a periodic orbit γ_0, then $\omega(\xi)$ is a periodic orbit.*

Proof. If $\omega(\xi) \setminus \gamma_0$ is not empty, then the connectedness of $\omega(\xi)$ implies the existence of a sequence $\xi_k \in \omega(\xi) \setminus \gamma_0$ and a $\xi_0 \in \gamma_0$ such that $\xi_n \to \xi_0$ as $k \to \infty$. If Σ is a transversal to f at ξ_0, then it follows from Lemma 11.2 that there exist a sequence $\{ t_k \}$ such that $\varphi^{t_k}(\xi_k) \in \Sigma \cap \omega(\xi)$ and $\varphi^{t_k}(\xi_k) \to \xi_0$ as $k \to \infty$. Since all of these points belong to $\omega(\xi)$ and Lemma 11.3 implies that $\omega(\xi)$ can intersect Σ in at most one point, we obtain a contradiction.

Theorem 11.2. *(Poincaré Bendixson Theorem) For the planar system (6.1), if $\gamma^+(\xi)$ is bounded and $\omega(\xi)$ contains no critical points, then $\omega(\xi)$ is a periodic orbit and is either equal to $\gamma^+(\xi)$ or equal to $\overline{\gamma^+(\xi)} \setminus \gamma^+(\xi)$. The same conclusion holds for $\gamma_-(\xi)$ and $\alpha(\xi)$.*

Proof. The set $\omega(\xi)$ is a nonempty compact invariant set which contains no critical points. Lemmas 11.5 and 11.6 imply that $\omega(\xi)$ is a periodic orbit. The other assertions are clear.

Theorem 11.3. *For the planar system (6.1), if $\gamma^+(\xi)$ is a bounded orbit, then one of the following is satisfied:*
(i) $\omega(\xi)$ contains only critical points
(ii) $\omega(\xi)$ is a periodic orbit
(iii) $\omega(\xi)$ consists of equilibrium points and orbits whose α- and ω-limit sets are equilibrium points.

Proof. If $\omega(\xi)$ contains no regular points, then $\omega(\xi)$ contains only critical points and the first alternative holds. Now suppose that $\omega(\xi)$ contains regular points of f. If all points of $\omega(\xi)$ are regular, then Theorem 11.2 implies that $\omega(\xi)$ is a periodic orbit which is the second alternative above. The only remaining case is when all points in $\omega(\xi)$ are not regular. Suppose that $y \in \omega(\xi)$ is a regular point and $\omega(y)$ is not a periodic orbit. We remark that $\omega(y) \subset \omega(\xi)$ and claim that $\omega(y)$ contains points which are not regular. In fact, if $y_0 \in \omega(y)$ is a regular point and Σ is a transversal to f at y_0, then there would be a sequence $\varphi^{t_k}(y) \in \Sigma \cap \omega(\xi)$ such that $\varphi^{t_k}(y) \to y_0$ as $k \to \infty$. Lemma 11.3 implies that $\varphi^{t_k}(y) = y_0$ for all k and so $\omega(y)$ is a periodic orbit, which is a contradiction. Therefore, $\omega(y)$ can contain only critical points and the theorem is proved.

Exercise 11.1. Prove the following result: *A periodic orbit γ of a planar system (6.1) is a local attractor if and only if there is a neighborhood U of γ such that $\omega(\xi) = \gamma$ for every $\xi \in U$.*
Hint. Use a transversal to the periodic orbit.

Exercise 11.2. Prove the following result: *Suppose that γ_1, γ_2 are two periodic orbits of a planar system (6.1) with γ_2 in the interior of γ_1 and let U be the annular set between γ_1 and γ_2. If there are no critical points or periodic orbits in U and $\xi \in U$, then $\omega(\xi)$ is either γ_1 or γ_2. Furthermore, if $\omega(\xi_0) = \gamma_1$ (resp. γ_2) for an $\xi_0 \in U$, then, for each $\xi \in U$, $\omega(\xi) = \gamma_1$ (resp. γ_2).*
Hint. Use the Poincaré Bendixson Theorem.

It is possible to prove the Brouwer fixed point theorem for C^1-mappings in the plane by using the Poincaré Bendixson Theorem and some of the above ideas.

Theorem 11.4. *(Brouwer's Fixed Point Theorem) If $\bar{B} = \{ x \in \mathbb{R}^2 : x^* x \le 1 \}$ and $g : \bar{B} \to \bar{B}$ is a C^1-mapping, then g has a fixed point in $\bar{B}$.*

Proof. Let $f(x) = g(x) - x$ and consider the differential equation $\dot{x} = f(x)$. For any $x \in \bar{B}$, we have $f(x)^* x = |g(x)| \cos\theta \, - 1$, where θ is the angle between the vectors $g(x)$ and x. If g has a fixed point on the boundary of $\bar{B}$, then we have nothing to prove. Therefore, if we suppose that this is not the case, then $f(x)^* x < 1$ and the Poincaré Bendixson Theorem implies that B, the interior of $\bar{B}$ contains either a periodic orbit or a critical point of the differential equation. If there is no periodic orbit in B, then we have a critical point of f and thus a fixed point of g and the proof is finished. If there is a periodic orbit in B, we let $S = \{\gamma : \gamma \text{ is a periodic orbit or a critical point in } B\}$

and let $R(\gamma)$ be the interior of γ if γ is a periodic orbit and $R(\gamma) = \gamma$ if γ is a critical point of f. We define a partial order on S by inclusion. From Zorn's Lemma, we see that each nonempty subset of S contains a least element, say γ_0. If γ_0 is a critical point, we are finished. Otherwise, γ_0 is a periodic orbit and there is no periodic orbit in $R(\gamma_0)$. If there is no critical point in $R(\gamma_0)$, then the Poincaré Bendixson Theorem implies that $\omega(\xi) = \alpha(\xi) = \gamma_0$ for all $\xi \in R(\gamma_0)$. It is easy to see that this contradicts the monotonicity of the Poincaré map associated with a transversal to a point $\xi_0 \in \gamma_0$. This completes the proof.

The Poincaré Bendixson Theorem suggests a way to determine the existence of a nonconstant periodic solution of an autonomous planar differential equation. If we construct a bounded domain $D \subset \mathbb{R}^2$ which contains no critical points and is positively invariant under the flow, then we are assured that there is a periodic orbit. Furthermore, if we can conclude that there is only one periodic orbit in D, then we will know that it is an attractor for the orbits through points in D.

Exercise 11.3. Show that the differential equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_2(1 - x_1^2 - 2x_2^2)$$

has a periodic orbit.
Hint. Let $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ and show that the derivative of this function along the solutions of the differential equation is ≥ 0 if $x_1^2 + x_2^2 < \frac{1}{2}$ and is ≤ 0 if $x_1^2 + x_2^2 > 1$.

We now give an important application of the Poincaré Bendixson Theorem to the van der Pol equation.

Theorem 11.5. *The equation of van der Pol*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \lambda(1 - x_1^2)x_2 \end{aligned} \tag{11.1}$$

has a nontrivial periodic orbit for all values of the scalar parameter λ.

Proof. (Yeh [1986]) We will consider the case $\lambda > 0$. When $\lambda = 0$, the equation becomes the linear harmonic oscillator. The case $\lambda < 0$ can be reduced to the first case by reversing time. Let us observe first that the origin is the only critical point of the vector field. Furthermore, it is easy to verify that the eigenvalues of the linearized equations at the origin have positive real parts. Therefore, no orbit with nonzero initial data can have its ω-limit set contain the origin. (The verification of this intuitive obvious fact will follow from the principle of linearization, Theorem 2.7.7, in the next chapter by replacing t by $-t$.) Thus, to show the existence of a periodic orbit, we will construct a Jordan curve $\mathcal{C}$ encircling the origin which will form the outer boundary of a positively invariant region and then invoke the Poincaré Bendixson Theorem to conclude that there is a periodic orbit in this region.

The idea for constructing $\mathcal{C}$ is to begin at a point A on the negative x_2-axis and use the special properties of the vector field to piece together various curve segments to obtain a curve lying in the left half plane which intersects the positive x_2-axis at a point E and such that the angle between the tangent vector to this curve and the vector field (11.1) is in the interval $(0, \pi)$. Since (11.1) is symmetric with respect to the origin, we can also define the reflection of this curve through the origin and obtain points A' and E'. If $A' > E'$, then the curve $AEA'E'A$ will be our Jordan curve $\mathcal{C}$.

First, we draw an auxiliary curve $\mathcal{Q}$,

$$\mathcal{Q}(x_1, x_2) \equiv -x_1 + \lambda(1 - x_1^2)x_2 = 0, \tag{11.2}$$

which has three components and their asymptotes given by $x_1 = \pm 1$ and $x_2 = 0$; see Figure 11.1. The component of this curve with asymptotes $x_2 = 0$ and $x_1 = -1$ is crossed from left to right by the vector field (11.1).

Figure 11.1.

To construct the first piece of $\mathcal{C}$, we take a point $A = (0, x_2^0)$ on the negative x_2-axis sufficiently far away from the origin, and follow the orbit of the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \lambda(1 - x_1^2)x_2 \end{aligned} \tag{11.3}$$

passing through the point A. An easy integration shows that this orbit intersects the line $x_1 = -1$ at the point $B = (-1, -2\lambda/3 + x_2^0)$. Along the arc AB we have

$$\frac{-x_1 + \lambda(1 - x_1^2)x_2}{x_2} - \frac{\lambda(1 - x_1^2)x_2}{x_2} = -\frac{x_1}{x_2} < 0;$$

thus, the crossing of AB by an orbit of (11.1) must be from right to left.

Next, we follow the orbit of the differential equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned} \tag{11.4}$$

emanating from B until the orbit hits the component of the curve (11.2) in the upper left quadrant. If the point A is taken sufficiently far from the origin, such a point of intersection will exist and we denote it by C. Along the curve BC we have

$$\frac{-x_1 + \lambda(1 - x_1^2)x_2}{x_2} + \frac{x_1}{x_2} = \lambda(1 - x_1^2) < 0$$

and, thus, the orbits of (11.1) cross BC from left to right.

For the next piece of $\mathcal{C}$, we first study the points of tangency of an orbit of the differential equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \lambda x_2 \end{aligned} \tag{11.5}$$

with the component of the curve (11.2) on the upper left quadrant. By implicitly differentiating (11.2), it is not difficult to verify that the first coordinate x_1^1 of such a point $D = (x_1^1, x_2^1)$ satisfies the equation

$$1 + (1 - \lambda^2)x^2 + 2\lambda^2 x^4 - \lambda^2 x^6 = 0. \tag{11.6}$$

When $x = -1$, the left side of (11.6) is positive, and when $|x|$ is sufficiently large the left side is negative; hence, there exists a solution x_1^1 of (11.6). Among the solutions of (11.6) we take the one nearest to -1. When the point A is sufficiently far from the origin, the point D lies to the right of C. It is clear that the orbits of (11.1) are crossing the segment of the curve (11.2) between C and D from left to right.

To continue $\mathcal{C}$, we follow the orbit of (11.5) starting from the point D until it hits the x_2-axis at a point which we denote by E. Since

$$\frac{-x_1 + \lambda(1 - x_1^2)x_2}{x_2} - \frac{-x_1 + \lambda x_2}{x_2} = -\lambda x_1^2 < 0,$$

the orbits of (11.1) cross the curve DE from left to right.

The first half of $\mathcal{C}$ is defined to be $ABCDE$. Let $A'B'C'D'E'$ be the reflection of $ABCDE$ with respect to the origin. Since D, hence E, is fixed, we may ensure that A' lies above E by taking A far from the origin. Also, observe that orbits of (11.1) cross the curve segment EA' from left to right. From the symmetry, it is now clear that the region encircled by the closed curve $\mathcal{C} = ABCDEA'B'C'D'E'A$ is positively invariant for the flow of (11.1). This completes the proof of the theorem.

Exercise 11.4. The Lienard form of the van der Pol equation is

$$\dot{x}_1 = x_2 - \lambda(\frac{1}{3}x_1^3 - x_1), \quad \dot{x}_2 = -x_1,$$

where $\lambda > 0$. Show that the periodic orbit must lie in the exterior of the disk $x_1^2 + x_2^2 < 3$ for every $\lambda > 0$.
Hint. Replace t by $-t$, let $V(x_1, x_2) = x_1^2 + x_2^2$ and use Theorem 8.2.

In Chapter 3, we will prove there is a unique periodic orbit of van der Pol's equation which is a global attractor in $\mathbb{R}^2 \setminus \{0\}$ when $\lambda > 0$, and unstable when $\lambda < 0$.

Let us give another application of the Poincaré-Bendixson Theorem to a problem in bifurcation. More details about this type of bifurcation will be given in a later section.

Example 11.1. Consider the planar system

$$\dot{x} = Ax + f(x), \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{11.6}$$

where $f(0) = 0$, $D_x f(0) = 0$. If we suppose that the origin is a local attractor, then Theorem 8.7 implies that there is a neighborhood U of $0 \in \mathbb{R}^2$ and a function $V \in C^1(U, \mathbb{R})$ such that V is positive definite and $\dot{V}$ is negative definite on U. Fix $c > 0$ and consider the level set $V^{-1}(c) = \{x \in U : V(x) = c\}$. If c is sufficiently small, then $V^{-1}(c)$ is a closed curve with the property that, for every $x \in V^{-1}(c)$, the vector field $Ax + f(x)$ points into the interior of $V^{-1}(c)$. Now, consider a C^1-small perturbation F of f such that, for every $x \in V^{-1}(c)$, the vector field $Ax + F(x)$ points into the interior of $V^{-1}(c)$. Also, assume that $F(0) = 0$ and the vector field $Ax + F(x)$ has only the origin as equilibrium point in the interior of $V^{-1}(c)$. If we further suppose that the origin of $Ax+F(x)$ is unstable, then the Poincaré-Bendixson theorem implies that there must be a periodic orbit in U. By varying the vector field f to the vector field F, a periodic orbit appeared through bifurcation.

As another illustrataion of the methods that we have been discussing in this chapter, we consider in some detail the *Esaki Diode*. If we consider an Esaki diode with the characteristic function $f(v)$ representing the current flow as a function of the voltage drop v, then Kirchoff's laws imply that the relation between the current I and the voltage v is given by

$$\begin{aligned} L\frac{di}{dt} &= E - Ri - v \equiv I(i, v), \\ -C\frac{dv}{dt} &= f(v) - i \equiv V(i, v), \end{aligned} \tag{11.7}$$

where E, R, C, L are respectively the applied voltage, the resistance, the capacitance and the inductance in the circuit. We assume that $vf(v) \geq 0$ for all v.

Lemma 11.7. *If there is an $A > 0$ such that $xf(x) > E^2/R$ for $|x| > A$, then there is a global attractor $\mathcal{A}$ for (11.7).*

Proof. If $W(i, v) = (Li^2 + Cv^2)/2$, then

$$\dot{W} = -[Ri(i - \frac{E}{R}) + vf(v)].$$

Let $W_0 = W(E/R, A)$. If $W(i, v) > W_0$, then either $|i| > E/R$ or $|v| > A$. If $|i| > E/R$, then $\dot{W} < 0$ and, if $|i| \leq E/R$, $|v| > A$, then

$$\dot{W} < -[Ri^2 - Ei + \frac{E^2}{R}] = -[Ri^2 - E(i - \frac{E}{R})] \leq -Ri^2 \leq 0.$$

For $i = 0$, $|v| > A$, we have also $\dot{W} < 0$. Therefore, $\dot{W} < 0$ in the region $W(i,v) > W_0$. This implies that the ω-limit set of any bounded positive orbit must lie in the set $\{(i,v) : W(i,v) \leq W_0\}$. Since the region $W < \rho$ is bounded for any $\rho > 0$ and $W(i,v) \to \infty$ as $|(i,v)| \to \infty$, it follows that every solution of (11.7) is bounded for $t \geq 0$. The conclusion of the lemma now follows from Theorem 7.3.

We now determine some of the properties of the global attractor when we impose more conditions on f.

Lemma 11.8. *If the conditions of Lemma 11.7 are satisfied and $f'(v) > 0$ for all v, then the global attractor is a singleton, which of course is an equilibrium point.*

Proof. First of all, it is clear that the hypotheses imply that there is a unique equilibrium point (i_0, v_0). Also, the eigenvalues of the linearization about (i_0, v_0) have negative real parts and so (i_0, v_0) is a local attractor. If we show that every solution of (11.7) approaches (i_0, v_0) as $t \to \infty$, then we will have proved the statement of the lemma. To prove this latter fact, let

$$Q(i,v) = \frac{1}{2L} I^2(i,v) + \frac{1}{2C} V^2(i,v). \tag{11.8}$$

Then $\dot{Q}(i,v) = -(RL^{-2}I^2(i,v) + f'(v)V^2(i,v)) \leq 0$ for all i, v since $f'(v) > 0$ for all v. Also, $\dot{Q} = 0$ if and only if $I(i,v) = 0 = V(i,v)$; that is, at the equilibrium point (i_0, v_0). Since Lemma 11.7 implies that each solution of (11.7) is bounded, it follows from the invariance principle (Theorem 8.3) that each solution of (11.7) approaches (i_0, v_0). The proof of the lemma is complete.

The most interesting cases in the applications are when f' changes sign and, in this case, several different types of situations arise.

Lemma 11.9. *If $-f'(v) < R^{-1}$ for all v, then there is only one equilibrium point of (11.7). If, in addition, if $\max_v(-f'(v)/C) > R/L$, then there is a value of E such that (11.7) has at least one periodic orbit.*

Proof. It is clear that there is a unique equilibrium point (i_0, v_0) and that one can choose E so that $(-f'(v_0)/C) > R/L$. From Lemma 11.7 (actually the proof of Lemma 11.7), we have seen that there is an ellipse Ω with center $(0,0)$ such that the orbits of (11.7) cross Ω from the outside to the inside. If we linearize about (i_0, v_0), it is easy to verify that the eigenvalues of the corresponding system have positive real parts. Now, using the same argument as in Example 11.1, we see that there is a periodic orbit in Ω and the lemma is proved.

More detailed results are know for this equation. In particular, it is possible to prove the following result which covers situations where there may be more than one equilibrium point of (11.7).

Lemma 11.10. *If there is an $A \geq 0$ such that $vf(v) \geq 0$ for all v, $vf(v) > E^2/R$ for $|v| > A$ and*

$$\frac{f'(v)}{C} + \frac{R}{L} > 0 \text{ for all } v,$$

then not only is there a global attractor $\mathcal{A}$ of (11.7), but the ω-limit set of each orbit belongs to the set of equilibrium points.

Proof. If we let

$$U(v) = \frac{(E-v)^2}{2R} + \int_0^v f(s)ds,$$

$$P(i,v) = -\frac{I^2}{2R} + U(v),$$

then (11.7) can be written as

$$\begin{aligned} L\frac{di}{dt} &= \frac{\partial P}{\partial i} \\ -C\frac{dv}{dt} &= \frac{\partial P}{\partial v}. \end{aligned} \tag{11.9}$$

Let $Q(i,v)$be defined as in (11.8), $S = Q + \lambda P$, where

$$-\frac{f'(v)}{C} < \lambda < \frac{R}{L}.$$

Then some rather lengthy calculations show that

$$\dot{S} = -[(R - \lambda L)L^{-2}I^2 + (f' + \lambda C)C^{-2}V^2] \leq 0$$

by our choice of λ. Furthermore, $\dot{S} = 0$ if and only if $I = 0 = V$; that is, the set of equilibrium points of (11.7). Since all solutions of (11.7) are bounded by Lemma 11.7, the conclusion of the lemma follows from the invariance principle (Theorem 8.3).

It also is useful to have methods for determining when certain regions do not have periodic orbits.

Theorem 11.6. *(Bendixson's Criterion) Let D be a simply-connected open subset of $\mathbb{R}^2$. If $\operatorname{div} \mathbf{f} \equiv \partial f_1/\partial x_1 + \partial f_2/\partial x_2$ is of constant sign and not identically zero in D, then $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has no periodic orbit lying entirely in the region D.*

Proof. If γ is a periodic orbit in D, then $f_1\,dx_2 - f_2\,dx_1 = 0$ on γ. Since the interior U of γ is simply-connected, we can apply Green's theorem to obtain

$$0 = \oint_\Gamma (f_1\,dx_2 - f_2\,dx_1) = \iint_U \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1\,dx_2.$$

This is a contradiction since our hypothesis implies that the integral on the right cannot be zero.

An easy but important generalization of the criterion of Bendixson is the following theorem:

Theorem 11.7. *(Dulac's Criterion) Let $D \subset \mathbb{R}^2$ be a simply-connected open set and $B(x_1, x_2)$ be a real-valued C^1-function in D. If the function* $\operatorname{div} B\mathbf{f} = \partial(Bf_1)/\partial x_1 + \partial(Bf_2)/\partial x_2$ *is of constant sign and not identically zero in D, then $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has no periodic orbit lying entirely in the region D.*

Exercise 11.5. Show that the equation

$$\dot{x}_1 = x_1 + x_2^3 - x_1 x_2^2, \quad \dot{x}_2 = 3x_2 + x_1^3 - x_1^2 x_2$$

has no periodic orbit in the region $x_1^2 + x_2^2 \le 4$.

Exercise 11.6. Consider the equation

$$\dot{x}_1 = -x_2 + x_1^2 - x_1 x_2, \quad \dot{x}_2 = x_1 + x_1 x_2\,.$$

Show that Bendixson's criterion will not preclude the existence of a periodic orbit and yet Dulac's criterion will for the function $B(x_1, x_2) = (1 + x_2)^{-3}(-1 - x_1)^{-1}$. *Hint.* The function div Bf is not of fixed sign but notice that the line $x_2 = -1$ is invariant and so no periodic orbit can intersect this line.

The Poincaré-Bendixson Theorem tells us that the only minimal sets in planar differential equations are equilibrium points and periodic orbits. In applications, it is necessary to know much more in order to determine the properties of the flow defined by the differential equation. In the case of the van der Pol equation, the flow is very simple. There is a unique periodic orbit to which every orbit, except for the equilibrium point, converges.

It is important to understand more about the number of equilibrium points and limit cycles (isolated periodic orbits) that a given differential equation may have. This is a very difficult problem. For polynomial vector fields, upper bounds for the number of equilibrium points are known. These can be given in terms of the degrees of the polynomials. The determination of bounds for the number of limit cycles is the celebrated *Hilbert's 16th problem.* Only recently (see Ilyashenko [], Écale []) has it been shown that, for a given polynomial vector field, the number of limit cycles is finite. At the present time, there is no upper bound known in terms of the degree of the polynomial vector field. It is even very difficult to estimate the number of limit cycles that can appear from the origin for a differential equation which is perturbation of the linear oscillator,

$$\begin{aligned} \dot{x} &= y + f_\lambda(x, y) \\ \dot{y} &= -x + g_\lambda(x, y) \end{aligned},$$

where (f_l, g_λ) are given polynomials of degree (m, n) which vanish for $\lambda \in \mathbb{R}^k$ equal to zero. Of course, it is always finite. In the case where $(m, n) = (2, 2)$, it is know (see Bautin[]) that the maximal number of limit cycles near zero is 3 and that there is a perturbation of the vector field that will give 3 limit cycles. Such information is not available when $(m, n) = (3, 3)$, for example. Examples have been constructed for latter case where there are a very large number of limit cycles (see Lloyd[]).

Some special cases of this latter problem are related to another problem that we have discussed in Section 1.8.1; namely, periodic solutions of scalar differential equations with periodic coefficients. In fact, consider the equation

$$\begin{aligned} \dot{x} &= \lambda x + y + P(x, y) \\ \dot{y} &= -x + \lambda y + Q(x, y) \end{aligned}, \tag{11.7}$$

where P, Q are homogeneous polynomials of degree n. If we introduce polar coordinates (r, θ), we have

$$\dot{r} = \lambda r + f(\theta) r^n, \quad \dot{\theta} = -1 + g(\theta) r^{n-1}.$$

If we write

$$\rho = r^{n-1}(1 - r^{n-1} g(\theta))^{-1},$$

then ρ satisfies the differential equation

$$\dot{\rho} = A(\theta)\rho^3 + B(\theta)\rho^2 - \lambda(n-1)\rho, \tag{11.8}$$

where

$$A(\theta) = -(n-1)g(\theta)(f(\theta) + \lambda g(\theta)), \quad B(\theta) = g'(\theta) - (n-1)(1\lambda g(\theta) - f(\theta)).$$

Limit cycles of (11.7) are mapped to isolated periodic solutions of the scalar equation (11.8). For a discussion, see Lins Neto [], Carbonell and Llibre [], Lloyd [].

The more general problem

$$\dot{z} = p_N(t) z^N + \ldots + p_1(t) z + p_0(t),$$

where z is a complex scalar and the vector $P(t) = (p_0(t), p_1(t), \ldots, p_N(t))$ is continuous and periodic in t, has been discussed by Lloyd[] and J. Devlin [Periodic solutions of polynomial non-autonomous differential equations. *Proc. Royal Soc. Edinburgh 123A* (1993), 783-801], where all of the other references can be found. They discuss the changes in the number of periodic solutions as the vector function $P(\cdot)$ varies.

æ