Chapter 2

Linear and Perturbed Linear Systems

One of the important topics in differential equations is concerned with the local theory; that is, the behavior of solutions near a given solution $\varphi^t(\xi)$. If we let $x = \varphi^t + y$, then the discussion reduces to determining properties of solutions of a nonautonomous differential equation near the equilibrium solution $y = 0$. As a first step, it is natural to replace the vector field by the first term in the Taylor series near 0. This leads to the study of linear differential equations, which we do in this chapter. In addition to giving basic properties of solutions of linear equations, we investigate, in some elementary situations, the effects of perturbations on stability.

2.1. General properties.

If $A \in C^0(\mathbb{R}, \mathbb{R}^{d\times d})$ (or $A \in C^0(\mathbb{R}, \mathbb{C}^{d\times d})$) is a $d \times d$ matrix function, we consider the linear equation

$$\dot{x} = A(t)x\,. \tag{1.1}$$

For any $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d$, Theorem 1.3.1 implies that there is a unique solution $x(t, \tau, \xi)$ through (τ, ξ) which is continuous in its arguments and Exercise 1.4.4 ensures that this solution exists for all $t \in (-\infty, \infty)$. Also, from the uniqueness and the fact that the vector field in (1.1) is linear in x, it follows, as in the discussion of linear autonomous equations in Section 1.5, that there is a $d \times d$ matrix function $X(t, \tau)$, $X(\tau, \tau) = I$, the identity, such that

$$x(t, \tau, \xi) = X(t, \tau)\xi\,. \tag{1.2}$$

Furthermore, each column of the matrix $X(t, \tau)$ is a solution of (1.1).

This leads to the following definition. A $d \times d$ matrix function $X(t)$ is said to be a *matrix solution of (1.1)* if each column of the matrix is a solution of (1.1). We write this as $\dot{X} = A(t)X$.

Proposition 1.1. *If $X(t)$ is a $d \times d$ matrix solution of (1.1), then either det $X(t) \neq 0$ for all t or det $X(t) = 0$ for all t.*

Proof. If there is a t_0 such that det $X(t_0) = 0$, then there is a nonzero vector $v \in \mathbb{R}^d$ such that $X(t_0)v = 0$. If $\psi(t) = X(t)v$, then $\psi(t)$ is a solution of (1.1) with $\psi(t_0) = 0$. Since 0 is a solution of (1.1), uniqueness of the solutions of the initial value problem imply that $\psi(t) = 0$ for all t and so det $X(t) = 0$ for all t.

A $d \times d$ matrix solution $X(t)$ of (1.1) is called a *fundamental matrix solution* if det $X(t) \neq 0$ for all t. A $d \times d$ matrix solution $X(t)$ of (1.1) is called a *principal matrix solution at τ* if $X(\tau) = I$. The matrix $X(t, \tau)$ in (1.2) is a principal matrix solution of (1.1) at τ. Furthermore, if $X(t)$ is a fundamental matrix solution of (1.1), then $X(t)X^{-1}(\tau)$ is a principal matrix solution at τ and

$$X(t, \tau) = X(t)X^{-1}(\tau) \tag{1.3}$$

for all t, τ.

A set $\{ x^1, x^2, \dots, x^d \}$ of vectors in $\mathbb{R}^d$ is said to be *linearly independent* if $\Sigma_{j=1}^d c_j x^j$ for any scalars c_j implies that $c_j = 0$ for all j. This set of vectors is said to be *linearly dependent* if they are not linearly independent.

Exercise 1.1. Prove that the set $\{ x^1, x^2, \dots, x^d \}$ of vectors in $\mathbb{R}^d$ are linearly independent if and only if $\det\,[x^1, x^2, \dots, x^d] \neq 0$.

In this terminology, we can say that $X(t)$ is a fundamental matrix solution of (1.1) if and only if the columns of the matrix $X(0)$ are linearly independent. Therefore, the determination of the behavior of solutions of the linear system is a finite dimensional problem. We select d linearly independent initial values at time 0, obtain the corresponding solutions of (1.1) and use (1.3) and (1.2).

Let A^* be the conjugate transpose of a matrix A. The *adjoint equation* of (1.1) is defined to be

$$\dot{y} = -A^*(t)y\,. \tag{1.4}$$

Proposition 1.2. *If $x(t)$ is a solution of (1.1) and $y(t)$ is a solution of (1.4), then $y(t)^*x(t)$ is a constant for all t.*

Proof. By differentiating the expression $y(t)^*x(t)$ with respect to t, we obtain $\dot{y}(t)^*x(t) + y(t)^*\dot{x}(t) = 0$ and thus $y(t)^*x(t)$ is a constant.

Proposition 1.3. *If $X(t)$ is a fundamental matrix solution of (1.1), then $Y(t) = [X(t)^{-1}]^*$ is a fundamental matrix solution of (1.4).*

Proof. If $Y(t)$ is a matrix solution of (1.4) with $Y(0) = [X(0)^{-1}]^*$, then Proposition 1.2 implies that $Y(t)^*X(t)$ is a constant for all t. Since this constant is the identity matrix, we have the conclusion that $Y(t) = [X(t)^{-1}]^*$.

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