1.4. Differential inequalities.

Let D_r denote the right hand derivative of a function. If $\omega(t, u)$ is a scalar function of the scalars t, u in some open connected set Ω, we say that a function $v(t)$, $a \le t < b$, is a *solution of the differential inequality*

$$D_r\, v(t) \le \omega(t, v(t)) \tag{4.1}$$

on $[a, b)$ if $v(t)$ is continuous and has a right hand derivative on $[a, b)$ that satisfies (4.1).

Theorem 4.1. *Let $\omega \in C^r(\Omega, \mathbb{R})$, $r \ge 1$, where $\Omega \subset \mathbb{R}^2$ is an open connected set. If $u(t)$ is a solution of the equation*

$$\dot{u} = \omega(t, u) \tag{4.2}$$

on $[a, b]$ and v is a solution of (4.1) on $[a, b)$ with $v(a) \le u(a)$, then $v(t) \le u(t)$ for $t \in [a, b)$.

Proof. For any positive integer n, let $u_n(t)$ designate the solution of the equation

$$\dot{u} = \omega(t, u) + \frac{1}{n}$$

with $u_n(a) = u(a)$. From Corollary 3.1 and Exercise 3.5, there is an n_0 such that u_n, for $n \ge n_0$, is defined on $[a, b]$ and $u_n(t) \to u(t)$ uniformly on $[a, b]$ as $n \to \infty$. Suppose that $v(t)$ is not $\le u(t)$ for $a \le t < b$. Then there exist $t_1, a < t_1 < b$, such that $v(t_1) > u(t_1)$. Since $u_n(t) \to u(t)$ uniformly on $[a, b]$ as $n \to \infty$, there is an integer n such that $v(t_1) > u_n(t_1)$. Thus, there is a $t_2 < t_1$ in (a, b) such that $v(t) > u_n(t)$ on $t_2 < t \le t_1$, $v(t_2) = u_n(t_2)$. This implies that

$$\begin{aligned} D_r\, v(t_2) \ge \dot{u}_n(t_2) &= \omega(t_2, u_n(t_2)) + \frac{1}{n} \\ &= \omega(t_2, v(t_2)) + \frac{1}{n} \\ &> \omega(t_2, v(t_2)), \end{aligned}$$

which is a contradiction. Consequently, $v(t) \le u(t)$ for $a \le t \le b$. This proves the theorem.

Corollary 4.1. *Suppose that $\omega(t, u)$ satisfies the conditions of Theorem 4.1 and, in addition, is nondecreasing in u. If u is a solution of (4.2) on $[a, b]$ and $v(t)$ is continuous and satisfies the integral inequality*

$$v(t) \le v_a + \int_a^t \omega(s, v(s))\, ds, \quad a \le t \le b, \quad v_a \le u(a)\,, \tag{4.3}$$

then $v(t) \le u(t)$, $a \le t \le b$.

Proof. If $V(t)$ is the right hand side of (4.3), then $v(t) \le V(t)$ and $\dot V(t) \le \omega(t, V(t))$, $V(a) = v_a \le u(a)$. Theorem 4.1 implies that $V(t) \le u(t)$ for $a \le t < b$. Since $V(t)$ is continuous on $[a, b]$, we have $V(t) \le u(t)$ for $a \le t \le b$, which proves the corollary.

Remark 4.1. If it not assumed that the function $\omega(t, u)$ in Corollary 4.1 is nondecreasing in u, then the conclusion in the corollary may not be true. The following example was supplied by X.-B. Lin. If $\omega(t, u) = -u$ and $u(0) = -1$, then $u(t) = -e^{-t}$. If $n \ge 2$ is an integer, then $v(t) = \frac{t}{n} - 1$ for $t \le n$ and $v(t) = 0$ for $t > n$ is a solution of the integral inequality (4.3) on $[0, \infty)$.

Corollary 4.2. *(The Gronwall Inequality) If α is a real constant, $\beta(t) \ge 0$ and $\varphi(t)$ are continuous real functions for $a \le t \le b$ which satisfy*

$$\varphi(t) \le \alpha + \int_a^t \beta(s)\varphi(s)\, ds, \quad a \le t \le b\,,$$

then

$$\varphi(t) \le \alpha e^{\int_a^t \beta(s)\, ds}, \quad a \le t \le b\,.$$

Proof. Apply Corollary 4.2 with $v_a = \alpha$, $\omega(t, u) = \beta(t)u$.

Corollary 4.3. *(Generalized Gronwall Inequality) If $\beta(t) \ge 0$, $\alpha(t)$ and $\varphi(t)$ are continuous real functions for $a \le t \le b$ which satisfy*

$$\varphi(t) \le \alpha(t) + \int_a^t \beta(s)\varphi(s)\, ds, \quad a \le t \le b\,,$$

then

$$\varphi(t) \le \alpha(t) + \int_a^t \beta(s)\alpha(s) e^{\int_s^t \beta(u)\, du}\, ds, \quad a \le t \le b\,.$$

If, in addition, $\dot\alpha(t)$ is continuous and $\dot\alpha \ge 0$, then

$$\varphi(t) \le \alpha(t) e^{\int_a^t \beta(s)\, ds}, \quad a \le t \le b\,.$$

Exercise 4.1. Prove Corollary 4.3. Let $R(t) = \int_a^t \beta(s)\varphi(s)\, ds$, obtain a differential inequality for R and find a solution of the inequality. If $\dot\alpha(t)$ is continuous, then integrate by parts.

Exercise 4.2. Consider the linear system of differential equations

$$\dot x = A(t)x + h(t)\,,$$

where the $d \times d$ matrix A and the d-vector h are continuous on an interval I, finite or infinite. Prove that the solution of the initial value problem exists on I. *Hint*: Fix a closed interval $\bar{I} \subset I$, take $\tau \in \bar{I}$, $\xi \in \mathbb{R}^d$ and let $v(t) = |x(t)|$. Obtain an integral inequality for v and use the generalized Gronwall inequality.

Differential inequalities are very convenient for obtaining bounds on the solutions of vector systems $\dot{x} = f(t, x)$. The inequality is obtained by differentiating scalar valued functions $V(t, x)$ along the solutions.

Exercise 4.3. For $x, y \in \mathbb{R}^d$, let $x \cdot y$ be the inner product of x and y. Suppose that $f \in C^r(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$, $r \geq 1$, and there exists a continuous function $\lambda \in C(\mathbb{R}, \mathbb{R})$ such that $x \cdot f(t, x) \leq -\lambda(t) x \cdot x$ for all t. For any $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^d$, show that the solution of the initial value problem exists for all t and satisfies the inequality

$$|x(t)| \leq e^{-\int_\tau^t \lambda(s)\, ds} |\xi|, \quad t \geq \tau\, .$$

Discuss the behavior of the solutions for $\lambda(t) \geq 0$. What happens if $\int_\tau^{+\infty} \lambda(s)\, ds = +\infty$? *Hint*: Let $V(x) = x \cdot x$ and find a differential inequality for $V(x(t))$ along the solution $x(t)$.

Exercise 4.4. Generalize the previous exercise to the case where $x \cdot Bf(t, x) \leq \lambda(t) x \cdot x$ where B is a positive definite symmetric matrix. *Hint*: Let $V(X) = x \cdot Bx$.

Exercise 4.5. Suppose that $|f(t, x)| \leq \lambda(t)|x|$ for all t, x and $\int_\tau^{+\infty} \lambda(s)\, ds < +\infty$. Show that each solution of $\dot{x} = f(t, x)$ approaches a constant as $t \to \infty$. If, in addition,

$$|f(t, x) - f(t, y)| \leq \lambda(t)|x - y|$$

for all t, x, y, show that there is a one-to-one correspondence between the initial positions and the limit values of the solution. Interpret the results for the linear equation $\dot{x} = A(t)x$ where the norm of the $d \times d$ matrix $A(t)$ is bounded by $\lambda(t)$.

Exercise 4.6. Suppose that $a(t)$ is a continuous scalar function, $\int_0^{+\infty} |a(s)|\, ds < \infty$. As in the previous exercise, show that the solutions of the equation $\dot{x} = -x + a(t)x$ have the form $x(t) = e^{-t} y(t)$, where $y(t) \to$ a constant as $t \to \infty$ and there is a one-to-one correspondence between the limits of the solutions and the initial position. Notice that you have shown that, for any constant c, there is a function $g(t) \to 0$ as $t \to \infty$ such that $x(t) = e^{-t}(c + g(t))$ is a solution of the differential equation. *Hint*: Find the differential equation for y.

Exercise 4.7. Consider the equation $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + a(t)x_1$, where a is the same function as in the previous exercise. Show that the solutions have the form

$$\begin{aligned} x_1(t) &= y_1(t) \cos t + y_2(t) \sin t \\ x_2(t) &= -y_1(t) \sin t + y_2(t) \cos t \end{aligned}$$

where $y(t) = (y_1(t), y_2(t)) \to$ a constant as $t \to \infty$ and there is a one-to-one correspondence between the limits of the solutions and the initial position. Comment about how this result relates the solutions to the solutions of the homogeneous equation $\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1$?

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