You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions. In your solution you can use only statements that were proven in class.

Name: ____________________

Question:	1	2	Total
Points:	30	30	60
Score:			

1. Consider the equation for $x \in \mathbb{R}^n$

$$\dot{x} = -x + g(x) \tag{1}$$

where

$$\|g(x)\| \leq C\|x\|^2.$$

for some $C > 0$.

Let $x(t)$ the soltion of eq.(1) such that $x(0) = x_0$.

(a) (10 points) Show that if $\|x_0\| \leq 1/C$ than $\|x(t)\| \leq \|x_0\|$ for every t.(**Hint:** write a differential equation for $\|x(t)\|$.)

Solution: We have

$$\frac{d}{dt}\|x(t)\| = \frac{(x(t) \cdot \dot{x}(t))}{\|x(t)\|} = -\|x(t)\| + \frac{(x(t) \cdot g(x(t))}{\|x(t)\|} \leq -\|x(t)\| + C\|x(t)\|^2$$

Thus if $\|x(t)\| \leq 1/C$ then $\frac{d}{dt}\|x(t)\| \leq 0$.

(b) (10 points) Use eq.(A4-4) page 253 from the book to show that, for $\epsilon \leq 1$, if $\|x_0\| \leq \epsilon/C$ than

$$\|x(t)\| \leq e^{-t}\|x_0\| + \epsilon \int_0^t e^{-(t-s)}\|x(s)\|ds.$$

Solution:
From eq.(A4-4) we have

$$x(t) = e^{-t}x_0 + \int_0^t e^{-(t-s)}g(x(s))ds$$

so that

$$\|x(t)\| \leq e^{-t}\|x_0\| + \int_0^t e^{-(t-s)}\|g(x(s))\|ds.$$

If $\|x_0\| \leq \epsilon/C$ than $\|x(t)\| \leq \epsilon/C$ and

$$\|g(x(s))\| \leq C\|x(s)\|^2 \leq \epsilon\|x(s)\|.$$

(c) (10 points) Conclude that forevery ϵ there exists δ such that

$$\|x(t)\| \leq e^{-(1-\epsilon)t}\|x_0\|$$

it $\|x(0)\| \leq \delta$. (**Hint:** Call $y(t) = e^t\|x(t)\|$. Use Gronwall Lemma and point 2) to show that $y(t) \leq y(0)e^{\epsilon t}$.)

Solution: Calling $y(t) = e^t\|x(t)\|$ and using point 2) we get

$$y(t) \leq y(0) + \epsilon \int_0^t y(s)ds$$

if $\|x(t)\| \leq \epsilon C$. From Gronwall Lemma we have

$$y(t) \leq y(0)e^{\epsilon t}$$

since $y(0) = \|x_0\|$ we have

$$\|x(t)\| \leq \|x_0\|e^{-(1-\epsilon)t}\|x(0)\|.$$

2. Consider the equation in $\mathbb{R}^2$

$$\dot{x} = E - \frac{(E \cdot x)}{(x \cdot x)} x \tag{2}$$

where $(x \cdot y) = x_1 y_1 + x_2 y_2$ and $E \in \mathbb{R}^2$.

(a) (10 points) Show that $H(x) = (x \cdot x)$ is an first integral for eq.(2).

Solution: We have

$$\dot{H}(x) = (\dot{x} \cdot \partial H(x)) = (E \cdot x) - \frac{(E \cdot x)}{(x \cdot x)}(x \cdot x) = 0.$$

(b) (10 points) Find all fixed point of eq.(2) and discuss their stability.

Solution: We immediately see from the equation that for x to be a fixed point we need that x is parallel to E. This condition is also sufficient so that x is a fixed point if and only if

$$x = \mu E$$

for some μ.

Moreover calling

$$\alpha(x) = \frac{(E \cdot x)}{(x \cdot x)}$$

we have

$$\frac{\partial \dot{x}}{\partial x} = -\begin{pmatrix} \alpha(x) + \partial_{x_1}\alpha(x)x_1 & \partial_{x_2}\alpha(x)x_1 \\ \partial_{x_1}\alpha(x)x_2 & \alpha(x) + \partial_{x_2}\alpha(x)x_2 \end{pmatrix}$$

so that for $x = \mu E$ we get

$$\frac{\partial \dot{x}}{\partial x} = -\frac{1}{\mu}\begin{pmatrix} 1 - \frac{E_1^2}{\|E\|^2} & -\frac{E_1 E_2}{\|E\|^2} \\ -\frac{E_1 E_2}{\|E\|^2} & 1 - \frac{E_2^2}{\|E\|^2} \end{pmatrix}$$

so that we have $\lambda_0 = 0$ is an eigenvalue with eigenvector E while $\lambda_1 = -1/\mu$ with eigenvector $E^\perp$. All fixed point are stable but not asymptotically stable.

(c) (10 points) Are there periodic orbit?

Solution: No. Every periodic orbit must be contained in a set of the form $H(x) = C$, that is a circle of radius $\sqrt{C}$. But each of this set contains two fixed point so it cannot contain a periodic orbit.