Write your name clearly on all sheets of paper you will turn in and possibly number them. Write clearly and large enough to be easily readable. Your poofs must be complete and clearly written.
Each question is worth the number of points indicated. Fifty five points will grant you an A but you have to solve at least one between ex. 1 and ex. 2.

1 (15 pts) Let $f : \mathbb{R} \to \mathbb{R}^2$ be a C^1 function (*i.e.* f is continuous, differentiable and the derivative is continuous). Prove that $\mu(f(\mathbb{R})) = 0$, where μ is the Lebesgue measure on $\mathbb{R}^2$.

2 Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two σ-algebras of subsets of X. Let μ_1 be a measure defined on $\mathcal{M}_1$ and μ_2 be a measure on $\mathcal{M}_2$.

a) (10 pts) Give a definition for the measure $\mu = \mu_1 + \mu_2$. On which σ-algebra $\mathcal{M}$ is μ defined? Show that your definition defines a measure.

b) (10 pts) Show that if μ_1 and μ_2 are complete than $\mu = \mu_1 + \mu_2$ is complete.

c) (10 pts) Show that if $\mu_1\big|_{\mathcal{M}}$ and $\mu_2\big|_{\mathcal{M}}$ are σ-finite than $\mu = \mu_1 + \mu_2$ is also σ-finite.

3 Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \mathcal{N})$ be a measurable space and $f : X \to Y$ a measurable function.

a) (10 pts) Show that $\nu(B) = \mu(f^{-1}(B))$ is a measure on $(Y, \mathcal{N})$.

b) (15 pts) Clearly if μ is finite than ν is finite. Is it true that if μ is σ-finite than also ν is σ-finite? Is it true that if μ is complete than ν is complete?

c) (20 pts) Assume now that if $X = Y = \mathbb{R}$, $\mathcal{M} = \mathcal{N} = \mathcal{L}$ and μ is the Lebesgue measure. Assume moreover that f is an increasing function with $\sup_{x\in\mathbb{R}} f = \infty$ and $\inf_{x\in\mathbb{R}} f = -\infty$. Show that ν is a Borel-Stieltjes measure. Construct the distribution function $F(x)$ of ν. Show that F is a distribution function. (**Hint** assume first that f is continuous and strictly increasing.)

4 Let δ_{x_0} be the measure defined by extending the premeasure μ_0 on the algebra generated by $\mathcal{A} = \{[a, b)|a, b \in \mathbb{R}\}$ by $\mu_0([a, b)) = 1$ if $a \le x_0 < b$ and 0 otherwise.

a) (10 pts) Show that δ_{x_0} is a Borel-Stieltjes measure. Find the σ-algebra $\mathcal{M}$ on which δ_{x_0} is complete and the distribution function of δ_{x_0}.

b) (5 pts) Show that any right continuous increasing function has at most a countable number of point of discontinuity.

c) (10 pts) Using point b) show that any Borel-Stieltjes measure μ can be written as

$$\mu = \bar{\mu} + \sum_{i=0}^{\infty} a_i \delta_{x_i}$$

where $\tilde{\mu}$ is generated by a continuous function G and $a_i \geq 0$, x_i are suitable numbers.

4 Let ν be a Borel-Stieltjes measure. We say that ν is absolutely continuous with respect to the Lebesgue measure μ if $\nu(E) = 0$ for every E with $\mu(E) = 0$ (*i.e.* every set with Lebesgue measure 0 has ν measure 0). We know that if the distribution function F of ν is Lipschitz than ν is absolutely continuous with respect to Lebesgue.

a) (5 pts) Show that it is enough to require that F is locally Lipschitz, *i.e* for every $K \subset \mathbb{R}$ compact there exists $C >)$ such that

$$|F(x) - F(y)| \leq C|x - y| \qquad \forall x, y \in K$$

b) (15 pts) The above condition is not necessary. Find a not locally Lipschitz function F that generates a measure ν absolutely continuous with respect to Lebesgue.

c) (Bonus) Can you give a necessary and sufficient condition on F to obtain a measure ν absolutely continuous with respect to Lebesgue? (**Hint**: a collection of interval with small Lebesgue measure must have a small ν measure)