2.5 n 10

We have

(I) $$\partial_{x_i} u_\lambda(x,t) = \lambda \frac{\partial}{\partial \lambda x_i} u(\lambda x, \lambda^2 t)$$

and $$\partial^2_{x_i} u_\lambda(x,t) = \lambda^2 \frac{\partial^2}{\partial(\lambda x_i)^2} u(\lambda x, \lambda^2 t)$$

while

$$\partial_t u_\lambda(x,t) = \lambda^2 \frac{\partial}{\partial(\lambda^2 t)} u(\lambda x, \lambda^2 t)$$

Thus

$$\partial_t u_\lambda(x,t) - \Delta u_\lambda(x,t) = \lambda^2\left(\frac{\partial}{\partial \tau} u(y,\tau) - \Delta_y u(y,\tau)\right) = 0$$

where $\tau = \lambda^2 t$ and $y = \lambda x$.

(II) Clearly we have

$$0 = \partial_\lambda\left(\partial_t u_\lambda(x,t) - \Delta u_\lambda(x,t)\right) = \partial_t\left(\partial_\lambda u_\lambda(x,t)\right) - \Delta\left(\partial_\lambda u_\lambda(x,t)\right)$$

so that $\partial_\lambda u_\lambda(x,t)$ solves the heat equation.

Observe that

$$v(x,t) = \partial_\lambda u_\lambda(x,t)\Big|_{\lambda=1}.$$

2.5 n 12

Define

$$u(x,t) = e^{-ct} v(x,t).$$

Substituting we get

$$-ce^{-ct} v(x,t) + e^{-ct} v_t(x,t) - e^{-ct} \Delta v(x,t) + c e^{-ct} v(x,t) = f$$

or

$$\begin{cases} v_t(x,t) - \Delta v(x,t) = e^{ct} f(x,t) & \text{on } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = g(x) \end{cases}$$

Thus

$$v(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) e^{cs} f(y,s) dy ds$$

so that

$$u(x,t) = \int_{\mathbb{R}^n} e^{-ct} \Phi(x-y,t) g(y) dy +$$

$$+ \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) e^{-c(t-s)} f(y,s) dy ds$$

2.5 n 13

Consider

$$v(x,t) = u(x,t) - g(t)$$

It solves

$$v_t - v_{xx} = -g'(t) \quad \text{on } \mathbb{R}^+ \times (0,\infty)$$

$$v = 0 \quad \text{on } \mathbb{R}^+ \times \{t=0\}$$

$$v = 0 \quad \text{on } \{x=0\} \times [0,\infty)$$

Let now

$$w(x,t) = \begin{cases} v(x,t) & x>0 \\ -v(-x,t) & x<0 \end{cases}$$

Reasoning as in ex 9 we find that w satisfies

$$\begin{cases} w_t - w_{xx} = \overset{h(x,t)}{\cancel{g'(t)}} & \text{on } \mathbb{R}^+ \times (0,\infty) \\ w = 0 & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

~~so that~~

$$\cancel{w(x,t) = \int_{\mathbb{R}} dy \int_0^t \Phi(x-y, t-s)\, g'(s)\, ds}$$

where

$$h(x,t) = \begin{cases} -g'(t) & x>0 \\ +g'(t) & x<0 \end{cases}$$

We thus have

$$w(x,t) = \int_{\mathbb{R}} \int_0^t \Phi(x-y, t-s)\, \cancel{g(s)}\, h(y,s)\, dy\, ds =$$

$$= -\int_{\mathbb{R}^+} \int_0^t \big(\Phi(x-y, t-s) - \Phi(x+y, t-s)\big)\, g'(s)\, dy\, ds$$

Integrating by part we get

$$w(x,t) = -\int_{\mathbb{R}^+} \int_0^t \frac{d}{dt}\big(\Phi(x-y, t-s) - \Phi(x+y, t-s)\big)\, g(s)\, ds\, dy$$

$$- g(t)$$

where we used that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} (\phi(x-y, \varepsilon) - \phi(x+y, \varepsilon))\, dy = 1$$

Thus we have

$$w(x,t) = -g(t) - \int_{\mathbb{R}^+} \int_0^t \Delta_y \big(\phi(x-y, t-s) - \phi(x+y, t-s)\big)\, g(s)\, ds\, dy$$

~~Integrating by part again~~ Thus we get

$$w(x,t) = -g(t) - \int_0^t \partial_y \big(\phi(x-y, t-s) - \phi(x+y, t-s)\big)\Big|_{y=0}^{y=\infty} g(s)\, ds =$$

$$-g(t) - 2\int_0^t \partial_x \phi(x, t-s)\, g(s)\, ds$$

So that we finally get, for $x > 0$

$$u(x,t) = w(x,t) + g(t) - 2\int \partial_x \phi(x, t-s)\, g(s)\, ds$$

That is the desired formula.

2.5 n 14

(a) Reapeting the proof of Theorem 3 pag 52 we see that

$$\phi'(r) = \frac{1}{r^{n+1}} \iint_{E(r)} \left(-4n u_s \psi - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i \right) dy\, ds$$

is valid for every $u(x,t)$ smooth.

We thus get

$$\phi'(r) \geq \frac{1}{r^{n+1}} \iint_{E(r)} \left(-4n \Delta u \, \psi - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i \right) dy\, ds$$

Proceding as on page 54 we get

$$\phi'(r) \geq 0$$

That, together with $\phi(0) = 4u(0,0)$

gives

$$\phi(r) \geq 4u(0,0)$$

That is exacty the thesis

(b) Suppose there exists $(x_0, t_0) \in U$ such that

$$v(x_0, t_0) = \max_{\bar{U}_T} v$$

Find r such that $E(x_0, t_0; r) \subset U_T$.

We get

$$v(x_0, t_0) \le \frac{1}{4r^n} \iint_{E(x_0, t_0; r)} v(y, s) \frac{|x_0 - y|^2}{|t - s|^2} \, dy \, ds$$

But since

$$\frac{1}{4r^n} \iint_{E(x_0, t_0; r)} \frac{|x_0 - y|^2}{|t - s|^2} \, dy \, ds = 1$$

we get

$$\frac{1}{4r^n} \iint_{E(x_0, t_0; r)} v(y, s) \frac{|x_0 - y|^2}{|t - s|^2} \, dy \, ds \le \max_{\bar{U}_T} v = v(x_0, t_0)$$

So that

$$v(x_0, t_0) = \frac{1}{4r^n} \iint_{E(x_0, t_0; r)} v(y, s) \frac{|x_0 - y|^2}{|t - s|^2} \, dy \, ds$$

From here you can proceed exactly
as in the proof of Th. 4 pag. 54

(c) We get

$$\partial_t \phi(u) = \phi'(u)\, u_t$$

while

$$\Delta \phi(u) = \phi''(u)\, Du \cdot Du + \phi'(u)\, \Delta u$$

So that

$$v_t - \Delta v = -\phi''(u)\, |Du|^2 \leq 0$$

(d) Observe That if u solve The heat eq. so do u_t and u_{x_i}.

Since $\phi(x) = x^2$ is convex we have That

$$u_t^2 \text{ and } u_{x_i}^2$$

are subsolution. ~~that is~~ But clearly The sum of subsolution is a subsolution.