No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ______________________________

Question:	1	2	3	4	5	6	Total
Points:	10	20	20	20	10	20	100
Score:							

Question 1 . . . . *10 point*

Consider the curve traced by the vector function

$$\mathbf{r}(t) = \cos(t^2)\mathbf{i} + \sin(t^2)\mathbf{j} + t^2\mathbf{k}$$

for $t > 0$. Compute $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$. Compute the curvature $\kappa(t)$.

Solution: We first compute the derivative

$$\dot{\mathbf{r}}(t) = -2t\sin(t^2)\mathbf{i} + 2t\cos(t^2)\mathbf{j} + 2t\mathbf{k}$$

so that we get

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{4t^2\sin(t^2)^2 + 4t^2\cos(t^2)^2 + 4t^2} = 2\sqrt{2}t$$

and

$$\mathbf{T}(t) = \frac{-\sin(t^2)\mathbf{i} + \cos(t^2)\mathbf{j} + \mathbf{k}}{\sqrt{2}}$$

We then have

$$\dot{\mathbf{T}}(t) = -\frac{2}{\sqrt{2}}t\cos(t^2)\mathbf{i} - \frac{2}{\sqrt{2}}t\sin(t^2)\mathbf{j}$$

and

$$\|\dot{\mathbf{T}}(t)\| = \frac{2}{\sqrt{2}}t$$

so that

$$\mathbf{N}(t) = -\cos(t^2)\mathbf{i} - \sin(t^2)\mathbf{j}.$$

Finally

$$\mathbf{B}(t) = \frac{\sin(t^2)\mathbf{i} - \cos(t^2)\mathbf{j} + \mathbf{k}}{\sqrt{2}}$$

For the curvature we get

$$\kappa(t) = \frac{\|\dot{\mathbf{T}}(t)\|}{\|\dot{\mathbf{r}}(t)\|} = \frac{1}{2}$$

Question 2 . . . . *20 point*

Consider the vector field

$$\mathbf{B}(x, y, z) = \frac{y}{x^2 + y^2}\mathbf{i} + \frac{-x}{x^2 + y^2}\mathbf{j}$$

and the curve traced by the vector function

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$$

(a) (10 points) Show that

$$\ddot{\mathbf{r}}(t) = \mathbf{B}(\mathbf{r}(t)) \times \dot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t))$$

Solution:
We have

$$\dot{\mathbf{r}}(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \mathbf{k}$$

and

$$\ddot{\mathbf{r}}(t) = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j}.$$

Since

$$\mathbf{B}(\mathbf{r}(t)) = \sin(t)\mathbf{i} - \cos(t)\mathbf{j}$$

we get

$$\mathbf{B}(\mathbf{r}(t)) \times \dot{\mathbf{r}}(t) = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j}.$$

(b) (10 points) Compute

$$\int_0^1 \mathbf{F}(\mathbf{r}(t))\mathrm{d}\mathbf{r}(t)$$

Solution: Since $\mathbf{F}(\mathbf{r}(t)) = \mathbf{B}(\mathbf{r}(t)) \times \dot{\mathbf{r}}(t)$ is always orthogonal to $\dot{\mathbf{r}}(t)$ we have

$$\int_0^1 \mathbf{F}(\mathbf{r}(t))\mathrm{d}\mathbf{r}(t) = 0.$$

Question 3 .... *20 point*

Consider the vector field

$$\mathbf{F}(x,y,z) = (xyz+1)e^{xyz}\mathbf{i} + x^2ze^{xyz}\mathbf{j} + x^2ye^{xyz}\mathbf{k}.$$

(a) (10 points) Show that $\mathbf{F}$ is conservative.

Solution: We call $P(x,y,z) = (xyz+1)e^{xyz}$, $Q(x,y,z) = x^2ze^{xyz}$ and $R(x,y,z) = x^2ye^{xyz}$. We compute curl $\mathbf{F}$:

$$\begin{aligned}
\frac{\partial R(x,y,z)}{\partial y} - \frac{\partial Q(x,y,z)}{\partial z} &= (x^2 + x^3yz)e^{xyz} - (x^2 + x^3yz)e^{xyz} = 0 \\
\frac{\partial P(x,y,z)}{\partial z} - \frac{\partial R(x,y,z)}{\partial x} &= (x^2y^2z + 2xy)e^{xyz} - (x^2y^2z + 2xy)e^{xyz} = 0 \\
\frac{\partial P(x,y,z)}{\partial y} - \frac{\partial Q(x,y,z)}{\partial x} &= (x^2yz^2 + 2xz)e^{xyz} - (x^2yz^2 + 2xz)e^{xyz} = 0
\end{aligned}$$

so that

$$\operatorname{curl}\mathbf{F}(x,y,z) = 0$$

and $\mathbf{F}(x,y,z)$ is conservative.

(b) (10 points) Find a potential function V for $\mathbf{F}$, that is find a function $V(x,y,z)$ such that grad $V(x,y,x) = \mathbf{F}(x,y,z)$

Solution: A general solution for

$$\frac{\partial V(x,y,z)}{\partial z} = x^2ye^{xyz}$$

is given by

$$V(x,y,z) = xe^{xyz} + \phi(x,y)$$

It is easy to see that $\phi \equiv 0$ gives a potential for $\mathbf{F}$.

Question 4 . . . . *20 point*

Assume the the Earth is a perfect sphere. The boundaries of the states of Colorado and Wyoming are "spherical rectangles". Colorado is bounded by the lines of longitude 102°W and 109°W and the lines of latitude 37°N and 41°N. Wyoming is bounded by the lines of longitude 104°W and 111°W and the lines od latitude 41°N and 45°N. Find the ratio between the surface area od Colorado and the surface area of Wyoming.(**Hint**: Assume that the radius of the Earth is R and project the two states on the equatorial plane.)

Solution: Assume that the radius of the Earth is R. The projection of the surface of the North Emisphere is the disk of radius R centered at the origin. The projection of a meridian of longitude $\phi°$ is the radius at an angle $2\pi\phi/360$ respect to the projection of the Greenwich meridian. Finally the projection of a parallel of latitude $\psi°$ is the cicle of radius $R\cos(2\pi\psi/360)$.

Thus, in polar coordinates (θ, r), Colorado project to

$$C = \{1.78 < \theta < 1.90,\ 0.75R < r < 0.80R\}$$

and Wyoming

$$W = \{1.81 < \theta < 1.93,\ 0.71R < r < 0.75R\}$$

while the area element on the surface of the earth is

$$dS = R\frac{r\mathrm{d}r\mathrm{d}\theta}{\sqrt{R^2 - r^2}}$$

We get

$$A(\text{Colorado}) = R\int_{0.75R}^{0.8R}\int_{1.78}^{1.90}\frac{r\mathrm{d}r\mathrm{d}\theta}{\sqrt{R^2 - r^2}} = R^2 0.12\left(\sqrt{1 - 0.75^2} - \sqrt{1 - 0.8^2}\right)$$

while

$$A(\text{Wyoming}) = R\int_{0.71R}^{0.75R}\int_{1.81}^{1.93}\frac{r\mathrm{d}r\mathrm{d}\theta}{\sqrt{R^2 - r^2}} = R^2 0.12\left(\sqrt{1 - 0.71^2} - \sqrt{1 - 0.75^2}\right)$$

Finally

$$\frac{A(\text{Colorado})}{A(\text{Wyoming})} = \frac{\sqrt{1 - 0.75^2} - \sqrt{1 - 0.8^2}}{\sqrt{1 - 0.71^2} - \sqrt{1 - 0.75^2}} = 1.44$$

Question 5 . . . . *10 point*

Let R be a region of area A, in the x, y plane, bounded by a simple closed curve C. Show that the coordinates of the centroid of the region are given by

$$\bar{x} = \frac{1}{A}\oint_C xy\,\mathrm{d}x + \frac{1}{A}\oint_C x^2\,\mathrm{d}y$$

(**Hint**: use Green's Theorem.)

Solution: If $\mathbf{F}(x, y) = xy\mathbf{i} + x^2\mathbf{j}$ then curl $\mathbf{F}(x, y) = x$ so that

$$\frac{1}{A}\oint xy\,\mathrm{d}x + \frac{1}{A}\oint_C x^2\ \mathrm{d}y = \frac{1}{A}\iint_R x\,\mathrm{d}x\mathrm{d}y = \bar{x}$$

Question 6 . . . . *20 point*

Let $\mathbf{F}(x, y, z) = 2xy^2z\mathbf{i} + 2x^2yz\mathbf{j} + (x^2y^2 - 6x)\mathbf{k}$ and S be the portion of the plane $y = z$ that lies inside the cylinder $x^2 + y^2 = 1$, oriented upward.

(a) (10 points) Compute

$$\iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S$$

without using Stoke's Theorem.

Solution: If $P(x, y, z) = 2xy^2z$, $Q(x, y, z) = 2x^2yz$ and $R(x, y, z) = x^2y^2 - 6x$ we have

$$\begin{aligned} \frac{\partial R(x,y,z)}{\partial y} - \frac{\partial Q(x,y,z)}{\partial z} &= 0 \\ \frac{\partial P(x,y,z)}{\partial z} - \frac{\partial R(x,y,z)}{\partial x} &= 6 \\ \frac{\partial Q(x,y,z)}{\partial x} - \frac{\partial P(x,y,y)}{\partial y} &= 0 \end{aligned}$$

so that $\text{curl}\,\mathbf{F} = -6\mathbf{j}$ while

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

Moreover we clearly have $\mathrm{d}S = \sqrt{2}\mathrm{d}x\mathrm{d}y$ so that

$$\iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n} = \iint_{x^2+y^2<1} 6\,\mathrm{d}x\mathrm{d}y = -6\pi$$

(b) (10 points) Compute

$$\iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\, \mathrm{d}S$$

using Stoke's Theorem. (**Hint**: since $\cos(\pi - t) = -\cos(t)$ we have that $\int_0^{2\pi} \cos^m(t)\sin^n(t) = 0$ if m is ... and n is ...)

Solution: Let C be the cuve traced by

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \sin(t)\mathbf{k}$$

fot $0 < t < 2\pi$. By Stoke's Theorem we have

$$\iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n} = \oint \mathbf{F}(\mathbf{r})\, \mathrm{d}\mathbf{r}$$

and since

$$\mathrm{d}\mathbf{r} = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \cos(t)\mathbf{k}$$

we get

$$\oint \mathbf{F}(\mathbf{r})\, \mathrm{d}\mathbf{r} = \int_0^{2\pi} (-2\cos(t)\sin^4(t) + 3\cos^3(t)\sin^2(t) - 6\cos^2(t))\, \mathrm{d}t$$

Observe that, since $\cos(\pi - t) = -\cos(t)$, we have

$$\int_0^{2\pi} \cos(t)\sin^4(t)\, \mathrm{d}t = \int_0^{2\pi} \cos^3(t)\sin^2(t)\, \mathrm{d}t = 0$$

so that

$$\oint \mathbf{F}(\mathbf{r})\, \mathrm{d}\mathbf{r} = -6\int_0^{2\pi} \cos^2(t))\, \mathrm{d}t = -6\pi$$

Useful Formulas

Geometry of curves:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \|\dot{\mathbf{r}}\| \qquad \mathbf{T} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \qquad \mathbf{N} = \frac{\dot{\mathbf{T}}}{\|\dot{\mathbf{T}}\|} \qquad \mathbf{B} = \mathbf{T} \times \mathbf{N} \qquad \kappa = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\|$$

Vector Fields: **F** is conservative on R if curl **F** = 0 and R is simply connected.

Integrals: If S is the surface described by $z = f(x, y)$ then

$$dS = \sqrt{1 + f_x^2 + f_y^2}\,\mathrm{d}x\mathrm{d}y$$

Stoke's Theorem:

$$\iint \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \oint \mathbf{F}(\mathbf{r})\mathrm{d}\mathbf{r}$$

Centroid of R:

$$\frac{1}{A}\iint_R x\,\mathrm{d}x\mathrm{d}y = \bar{x}$$

where A is the area of R