

Solutions for Practice Final A for Calculus I, Math 1501

I $\lim_{x \rightarrow 3} \frac{\sqrt{x-3}}{\sqrt{|x-3|}}$ does not exist. (The limit of the numerator is not zero, and the limit of the denominator is zero.)

$$\lim_{x \rightarrow 0} \frac{2^x - 1}{x} = \lim_{x \rightarrow 0} \frac{2^x - 2^0}{x} = \left. \frac{d}{dx} 2^x \right|_{x=0} = \ln 2$$

II

$$\frac{x^2 - (3 + A) + 3A}{x - 2} = x - (A + 1) + \frac{A - 2}{x - 2}$$

and this is continuous exactly for $A = 2$. The sine function is always continuous. So the answer is $A = 2$.

$$f(x) = \frac{\sqrt{2} \sin(x) - 1}{\sqrt{2}x - 1}$$

is discontinuous at $x = 1/\sqrt{2}$, where it is undefined, and there is nothing you can do about it.

III For part (a), you see exactly two points where the tangent to the curve has slope 1, so the answer is two.

For part (b), there is one point, at $(1, 1)$ where the slope is -1 .

To see this is right from the formula,

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx} 2xy$$

which means

$$y' = \frac{3x^2 - 2y}{3y^2 - 2x} \quad (*)$$

Together with $y' = -1$, this gives us

$$3(x^2 - y^2) = 2(y - x)$$

One solution of this is $y = x$. If $y \neq x$, we can divide by $y - x$ and get $3(x + y) = -2$ or

$$y = -\frac{2}{3} - x \quad (**)$$

Now use both of these in the original equation $x^3 + y^3 = 2xy$. If $y = x$, this reduces to

$$2x^3 = 2x$$

which is solved by $x = 0$ and $x = 1$. Clearly, at $x = 0, y = 0$ the slope is not -1 , so this point is out. (It is a spurious point; when $x = y = 0$, you can't see what the slope is in (*).) But at $x = 1, y = 1$ it is. Next, suppose (**) holds. Eliminating y from the original equation, we have

$$x^3 - \left(x + \frac{2}{3}\right)^3 = -2x \left(x + \frac{2}{3}\right)$$

which reduces to $8/27 = 0$ which is nonsense. The line (**) does not meet the curve at all. In fact, the line is an asymptote to the curve. so the final answer is that $(1, 1)$ is the only point on the curve with slope -1 .

IV

$$\tan(\theta) = \frac{100t}{500 + 75t}$$

so

$$\sec^2(\theta)\theta' = \frac{d}{dt} \frac{100t}{500 + 75t} .$$

At $t = 6$, this is $\theta' = 4/101$ radians per minute.

V If $ab = 1000$, then $b = 1000/a$. Let

$$f(a) = a + \frac{1000}{a}$$

for $a > 0$. The minimum is the critical point with $a = 10\sqrt{10}$. **which makes $a = b$. This is the key.**

The best we can do if a and b are to be integers is $a = 25, b = 40$ and $a = 40, b = 25$ and in either case, $a + b = 65$.

For a purely algebraic approach, observe that

$$4ab = (a + b)^2 - (a - b)^2$$

so

$$(a + b)^2 = 4ab + (a - b)^2 .$$

Given that $ab = 1000$, we have that

$$(a + b)^2 = 4000 + (a - b)^2 .$$

So again we see that we want a and b as close as possible and so 50 and 70 are the best choices.

Finally, $1000 = 2^3 5^3$, so there are 16 possible choices for a , so you could do this by listing all the choices, but that take a little while. The key here is to see that you want a and b to be as close to equal as possible, and that is what the calculus told us.

VI The area will be $4xy$ with

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 .$$

and $0 \leq x \leq 2$ and $0 \leq y \leq 3$. It will be easier to maximize x^2y^2 than the area directly; this avoids square roots. Let

$$f(x) = 9x^2 \left(1 - \frac{x^2}{4}\right) .$$

$f' = 0$ at $x = \sqrt{2}$ and $x = 0$ for $x \geq 0$. But

$$f(0) = f(2) = 0$$

so the maximum is at $x = \sqrt{2}$. This means that $y = 3/\sqrt{2}$ so

$$4xy = 12$$

is the maximum area.

VII The integral for the area is

$$A = \int_0^2 (2x - x^2) dx = 4/3 .$$

Next

$$\bar{x} = \frac{1}{A} \int_0^2 x(2x - x^2) dx = 1 .$$

Now, to do the y component using the similar formula, which illustrates breaking the integral into pieces, complete the square for the lower curve:

$$y = 2x^2 - 2x = 2(x - 1/2)^2 - 1/2$$

Solving this for x , we get

$$x = 1/2 \pm \sqrt{y/2 + 1/4} .$$

Thus

$$\bar{y} = \frac{1}{A} \left(\int_{-1/2}^0 2y\sqrt{y/2 + 1/4} dy + \int_0^2 y(1/2 + \sqrt{y/2 + 1/4} - \sqrt{y}) dy \right) =$$

$$\frac{29 + 25\sqrt{5} - 48\sqrt{2}}{30} \approx 0.5673 .$$

finally the volume can be done with Pappus's theorem:

$$\text{Vol} = 2\pi(1 + \bar{x})A = 16\pi/3$$

or by shells.

VIII

$$\int_0^{100} (200 - x/2)dx .$$

IX

$$\int_{\pi/4}^1 \sin^{-2}(x)dx = 1 - \cot(1) .$$

$$\int_0^1 \frac{x^2}{\sqrt{1+x}}dx = \ln(2) - 7/12 .$$

$$\int_0^1 \frac{1}{8 - x^2 - 2x}dx = \frac{11}{720} + \frac{\ln(5/2)}{108} .$$

X

$$\int_0^1 (2^x + x^2)dx = \frac{1}{\ln(2)} + \frac{1}{3} .$$

$$\int_0^1 x2^x dx = \frac{2}{\ln(2)} - \left(\frac{1}{\ln(2)} \right)^2 .$$

$$\int_0^1 \frac{1}{x^2 + 2x + 2}dx = -\frac{\pi}{4} + \arctan(2) .$$

XI $1/\sqrt{3}$ and $-(1/2 + i\sqrt{3}/2)$

XII $2^{1/8}e^{i\theta_k}$ where $\theta_k = \pi/16 + k\pi/2$ for $k = 0, 1, 2$ and 3 .