# Practice Test IB: Solutions

(Ia): Limit does not exist. As $x \to \frac{\pi}{2}$ the denominator tends to $\frac{\pi}{2}$ and $x \sin x$ also tends towards $\frac{\pi}{2}$. Since

$$\tan x = \frac{\sin x}{\cos x} \ ,$$

and since $\lim_{x\to\pi/2} \sin x = 1$ and $\lim_{x\to\pi/2} \cos x = 0$ the limit does not exist.

(Ib) $\frac{1}{2}$

(Ic) 6

(Id) 11

(IIa): We make a proof using induction.

a) Check first the statement for $n = 0$: $5 + 4^2 = 21$ which is divisible by 21.

b) Assume that the statement is true for some number $k$, i.e.,

$$5^{2k+1} + 4^{k+2} \ \text{ is divisble by } \ 21 \ .$$

We have to deduce from this that

$$5^{2(k+1)+1} + 4^{(k+1)+2} \ \text{ is divisble by } \ 21 \ .$$

This can be done as follows: Write

$$5^{2(k+1)+1} + 4^{(k+1)+2} = 25 \cdot 5^{2k+1} + 4 \cdot 4^{k+2}$$

$$= 25 \cdot (5^{2k+1} + 4^{k+2}) - 25 \cdot 4^{k+2} + 4 \cdot 4^{k+2} = 25 \cdot (5^{2k+1} + 4^{k+2}) - 21 \cdot 4^{k+2} \ .$$

the first number in parenthesis is divisible by 21 since $5^{2k+1} + 4^{k+2}$ is divisible by 21. This is precisely our assumption. The other number is obviously divisible by 21. Thus the whole number is divisible by 21. Thus, $5^{2n+1} + 4^{n+2}$ is divisible by 21 for all $n = 0, 1, 2, \ldots$.

(IIb): It is obvious that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

holds for $n = 1$.

Next we assume that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{k}} < 2\sqrt{k} \ ,$$

for some number $k$. Using this we have to show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1} \ .$$

Obviously,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{k+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \ ,$$

and therefore

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{k+1}} < 2\sqrt{k} + \frac{1}{\sqrt{k+1}} \ .$$

Therefore, all we have to show that

$$2\sqrt{k} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}.$$

Multiplying both sides by $\sqrt{k+1}$ yields

$$2\sqrt{k(k+1)} + 1 < 2(k+1)$$

or

$$2\sqrt{k(k+1)} < 2k+1 \ .$$

Squaring both sides yields

$$4k(k+1) < (2k+1)^2$$

which is certainly true since

$$1 < 4k + 1 \ .$$

(IIIa): Raise both sides to the third power. This reduces the question to showing that $a + b$ is less than $a + 3a^{2/3}b^{1/3} + 3a^{1/3}b^{2/3} + b$, which is obviously true since $a$ and $b$ are positive. The condition $a > b$ is not needed here.

(IIIb): Square both sides. This reduces the question to the following

$$\frac{a}{1+a} < \frac{b}{1+b} \ .$$

This problem is the same as showing that

$$a(1+b) < b(1+a) \ ,$$

which is the same as saying that $a < b$ which is true.

(IIIc): This statement is false. Take, e.g., $a = 5$ and $b = 1$.