Detailed Solutions for Practice Quiz IC

(I) (a) Finding all points with

$$|f(x) - 2| = 1 \qquad (*)$$

is equivalent to finding all solutions of $f(x) - 2 = \pm 1$, i.e., finding all x that satisfy either

$$f(x) = 3 \qquad \text{or} \qquad f(x) = 1\ .$$

(Getting rid of the absolute values turns one equation into two...) So we solve these. The first equation is solved by $x = 1/2$, and the second leads to $x + 2 = x + 1$ which has **no solution**.

(b) The point $x = 1/2$ found in (a) is a boundary point of the set in question. Any remaining boundary points would be points of discontinuity of f. (At a discontinuity, $|f(x) - 2|$ could "jump over" the value 1 without passing through it.) There is a discontinuity at $x = -1$, where we divide by zero, and that's all.

Thus, the boundary points are $x = 1/2$ and $x = -1$, and the intervals to be considered are

$$(-\infty, -1) \qquad (-1, -1/2) \qquad (-1/2, \infty)$$

These are intervals on which the inequality is uniformly true or false.

It remains to see which is which by testing any points. The inequality is clearly false anywhere near $x = -1$ because $f(x)$ is very large there, so only the third interval can work. Since $f|(2) - 2| = 0$, it is true in this whole interval. Final answer:

$$\text{for } \mathbf{x} > -\mathbf{1/2}$$

Finally, (c) is pretty similar: We first find al solutions of $f(x) - 2 = \pm 1/10$, and then find all points of discontinuity of f. These points together separate intervals on which the inequality is uniformly true of false.

First, $f(x) - 2 = 1/10$ is solved by $x = -1/11$, and $f(x) - 2 = -1/10$ is solved by $x = 1/9$. The point of discontinuity, $x = 1$ is the same as before. So this time the intervals to consider are

$$(-\infty, -1) \qquad (-1, -1/11) \qquad (-1/11, 1/9) \qquad (1/9, \infty)$$

and one easily sees as above that the inequality holds only in the third one. Final answer:

$$\text{for } \mathbf{x} \text{ in } \quad (-\mathbf{1/11}, \mathbf{1/9})$$

(II) (a) This limit is zero since $\lim_{x\to 0} x = \lim_{x\to 0} \sin x = 0$. So use the product theorem to put together the pieces.

(b) This limit is 1/4 since

$$\frac{(x-1)^2}{(x^2-1)^2} = \frac{1}{(x+1)^2}$$

which is a rational function continuous at $x = 1$ (So the limit can be computed by evaluation.)

(c) This limit is 3. One way to see this is using

$$x^3 - 1 = (x-1)(x^2 + x + 1) \ .$$

(d) This limit does not exist – the numerator is non-zero at $x = 1$ while the denominator has a zero there, and both the denominator and numerator are continuous.

(III) (a) this is continuous ony at $c = 0$ since one the one hand, $f(0) = 0$, and on the other hand

$$\lim_{x \to 0} f(x) = 0 \ .$$

But at any other value c there are values of x arbitrarily close to c with $f(x) > c^2/2$, and there are values of x arbitrarily close to c with $f(x) = 0$, restricting the inputs to f to be closer and closer to c does not bring the outputs closer and closer.

Final amswer: **Continuous only at $c = 0$.**

The above answer and expalnation avoids epsilons and deltas, but here is an alternate using them: Pick $\epsilon > 0$. To establish continuity at c, we need to find a formula $\delta(\epsilon)$ so that

$$|x - c| < \delta(\epsilon) \Rightarrow |f(x) - f(c)| < \epsilon \qquad (**)$$

If $c = 0$, the right hand side is $|f(x)| < \epsilon$, or just $f(x) < \epsilon$ since $f(0) = 0$ and f is a non-negative function. But by the definition,

$$f(x) \leq x^2$$

so if $x < \sqrt{\epsilon}$, then $f(x) \leq \epsilon$. Therefore, with $c = 0$, and $\delta(\epsilon) = \sqrt{\epsilon}$, (**) holds.

On the other hand, if $c \neq 0$, there are x values arbitrarily close to c with

$$|f(x) - f(c)| > c^2/2$$

so no positive $\delta(\epsilon)$ exists for any *epsilon* with $\epsilon < c^2/2$. (Note: dividing c^2 by two was arbitrary – the same argument works if you repalce $c^2/2$ by any number less than c^2.)

(b) In this case, $f(x)$ is undefined for $x = 2$, and is equal to 1 for all $x > 2$, and -1 for all $x < 2$. There is a jump at $x = 2$, so it is discontinuous there, but otherwise it is constant, which is certainly continuous.

Final amswer: **Continuous except at $c = 2$.**

(IV) (a) This is unbounded and monotone increasing. In fact, with $a_n = n^2/(n+1)$.

$$\frac{a_{n+1}}{a_n} = \frac{n^3 + 3n^2 + 3n + 1}{n^3 + 2n^2} > 1$$

for all n, which means $a_{n+1} > a_n$ for all n.

(b) This is unbounded, because of the growth of the factor n, and is not monotone because of the oscillations of the factor $\sin^2(n)$.

(c) This sequence which starts out as

$$\{1,\ \frac{5}{4},\ \frac{5}{4},\ \frac{17}{16},\ \ldots\}$$

is clearly not monotone, but it is bounded because it is decreasing after $n = 2$ and stays non-negative.

To see that it is decreasing after $n = 2$, let $a_n = (n^2+1)/2^n$ and compute

$$\frac{a_{n+1}}{a_n} = \frac{n^2+2n+2}{2n^2+2} < 1$$

for all $n > 2$. (The difference between the denominator and the numerator is $n^2 - 2n$ which is positive for $n > 2$. Hence the denominator is bigger for such n, and hence $\frac{n^2+2n+2}{2n^2+2} < 1$.) Therefore, from $n = 2$ onward, $a_{n+1} < a_n$.

(d) This sequence is bounded and monotone decreasing. To see this, let $a_n = \sqrt{n+1}/\sqrt{n}$ and compute

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{n^2+2n}}{\sqrt{n^2+2n+1}} < 1$$

Hence the sequence is decreasing, so the first term, $\sqrt{2}$, is an upper bound (in fact it is the least upper bound since it belongs to the sequence...) and all the terms are positive, so 0 is a lower bound. Hence the sequence is bounded. (Note that 0 is not the greatest lower bound, it is just an obvious lower bound. Almost as obvious is the fact that $a_n > 1$ for all n, and in fact 1 is the greatest lower bound. Since it is not a term in the sequence, to show this, one has to show that for any fixed $B > 1$, B is not a lower bound; i.e., there is some n with $a_n < B$. Squaring, this is the same as

$$\frac{n+1}{n} > B^2$$

which reduces after a bit of manipulation to

$$n > \frac{1}{B^2-1}$$

which is finite since $B > 1$, and hence $B^2 - 1 > 0$. So all we have to do is to take n to be the first integer larger than $1/(B^2-1)$ and we would have

$$a_n < B$$

so B cannot be a lower bound for $B > 1$, and thus 1 is the greatest lower bound.