Test IV for Calculus I, Math 1501, Nov. 23, 1999

I: (25 points) Consider the region Ω in the x–y plane that lies below the line $y = 1 + x$ and above the parabola $y = (x-1)^2$.

a) Compute the area of Ω.

Solution: The curves $y = x+1$ and $y = (x-1)^2$ meet at $x = 0$ and $x = 3$, so these are our limits of integration. The only reasonable wat to slice this region is vertically. The curve $y = x+1$ is on top, and the height at x is

$$h(x) = x + 1 - (x-1)^2 = 3x - x^2$$

Hence the area is

$$\int_0^3 h(x)\mathrm{d}x = \int_0^3 (3x - x^2)\mathrm{d}x = \frac{9}{2} \ .$$

b) Compute the volume of the solid obtained by rotating Ω about the x–axis.

Solution: Here we use "washers", and the volume is

$$V = \pi \int_0^3 ((x+1)^2 - (x-1)^4)\mathrm{d}x = \pi\frac{72}{5} \ .$$

c) Compute the volume of the solid obtained by rotating Ω about the y–axis.

Solution: Here we use shells.

$$V = 2\pi \int_0^3 x(3x - x^2)\mathrm{d}x = \pi\frac{27}{2} \ .$$

II: (20 points) A theater has a curtain that is 10 meters high, 20 meters wide, and has a mass density of 1 kg/m^2. How much work is done in raising the curtain to the level of its top edge? (Use the approximate value g $=$ 10m/s^2.)

Solution: Consider a strip of curtain of height dy a distance y down from the top. The weight (force) is 200dy Newtons, so the work in Newton–meters is

$$\int_0^{10} 200y\mathrm{d}y = 10,000 \ .$$

III: (25 points)
a) Consider the region in the positive quadrant of the x–y plane bounded by $x^2 + y^2 = 1$. Compute the centriod of this region.

Solution: Clearly the area is $pi/4$. The height of a vertiacal strip is $h(x) = sqrt1 - x^2$. So we compute

$$\int_0^1 xh(x)\mathrm{d}x = \frac{1}{3} \ .$$

Hence

$$\bar{x} = \frac{4}{3\pi}$$

and since $y = x$ is a line of symmetry,

$$\bar{y} = \bar{x} = \frac{4}{3\pi} \ .$$

b) Consider an equilateral triangle of unit side length centered at the point $(2,2)$ in the x–y plane. Compute the volume of the solid produced by rotating this triangle about the line $y = -x$.

Solution: Use Pappus's theorem. The distance from the centroid, $(2,2)$, to the line is clearly $2\sqrt{2}$. The area of an equilateral trianlge is $sqrt3/4\ell^2$ where ℓ is the side length. Here, that is 1, so

$$V = 2\pi(2\sqrt{2})\frac{\sqrt{3}}{4} = \pi\sqrt{6} \ .$$

c) Compute the surface area of the solid in part b).

Solution: By simmalr triangles, one finds that if the sides of the triangle are moved out a distance h, the new side length ℓ is

$$\ell = (1 + 2\sqrt{3}h)$$

As in part (b), the volume V_h of the thickend region is:

$$V_h = \pi\sqrt{6}(1 + 2\sqrt{3}h)^2$$

Differentiating this, we get the surface area S:

$$S = \pi\sqrt{6}4\sqrt{3} = 12\sqrt{2}\pi \ .$$

IV: (30 points) Evaluate the following integrals.

$$(a) \qquad \int_1^4 \frac{x^2}{2\sqrt{x} - 1} \mathrm{d}x$$

Solution: Use $u = 2\sqrt{x} - 1$ so that $x = (u + 1)^2/4$ and

$$\mathrm{d}x = \frac{u + 1}{2} \mathrm{du} \ .$$

When $x = 1$, $u = 1$ and when $x = 4$, $u = 3$ So the integral becomes:

$$\int_1^3 \frac{(u + 1)^5}{32} \frac{1}{u} \mathrm{d}x = \frac{1069}{120} + \frac{\ln(3)}{32} \ .$$

$$(b) \qquad \int \frac{\ln(x)}{x^3} \mathrm{d}x$$

Solution: By parts, one gets

$$-\frac{1}{4x^2} (2\ln(x) + 1) + C \ .$$

$$(c) \qquad \int \frac{1}{x(\ln(x))^3} \mathrm{d}x$$

Solution: Substituting $u = \ln(x)$, one gets

$$-\frac{1}{2\ln(x)^2} + C \ .$$

$$(d) \qquad \int_0^{\pi/2} \frac{\sin(x)}{\sqrt{1+\cos(x)}} \mathrm{d}x$$

Solution: Substituting $u = 1 + \cos(x)$, one gets

$$2(\sqrt{2} - 1) \ .$$