February 16, 2000

2601C2 Calculus III for CS
Spring 2000

SOLUTIONS TO MIDTERM EXAM I.

PROBLEM 1. (12 points)

(i) Find the equation in x, y, z of the plane that contains the point $P(1,1,1)$ and the line ℓ given by

$$\frac{x-1}{2} = y+1 = \frac{z-1}{-1}$$

(ii) Is the point $Q(-1,2,2)$ in this plane? If not, how far is it from the plane?

(iii) Is the point $Q(-1,2,3)$ in this plane? If not, how far is it from the plane?

SOLUTION: (i) Pick two points on the line, e.g. $S(3,0,0)$ and $T(5,1,-1)$, then compute the cross product $\mathbf{n} := \vec{SP} \times \vec{TP} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$. Hence the equation is $2(x-1) + 4(z-1) = 0$ or $x + 2z = 3$

(ii) Yes. Plug the coordinates $Q(-1,2,2)$ into the equation $x + 2z = 3$ to see that it is satisfied.

(iii.) No. The distance can be obtained as

$$\frac{|\vec{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{4}{\sqrt{20}} = \frac{2}{\sqrt{5}}$$

PROBLEM 2. (20 points) Let

$$A = \begin{pmatrix} 1 & 2 & 5 & 1 & 0 \\ -1 & -1 & -4 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 \\ 1 & 2 & 5 & 0 & -1 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

(i) Give all solutions to $A\mathbf{x} = \mathbf{c}$.

(ii) Find a basis in the column space of A.

(iii) Find a basis in the nullspace of A.

(iv) Characterize all vectors $\mathbf{b} \in \mathbf{R}^4$ such that $A\mathbf{x} = \mathbf{b}$ is solvable. Is this equation solvable for $\mathbf{b} = (1 \quad 1 \quad 1 \quad 1)^t$?

SOLUTION

(i) After full row reduction you get

$$\left(\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The full solution is

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

with s, t free parameters.

(ii) The basis in the column space is the pivot columns in A, i.e.

$$R(A) = Span\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

(iii). The basis in the nullspace is

$$N(A) = Span\left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

(iv.) Full row reduction with a general right hand side $\mathbf{b}$ gives

$$\left(\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 1 & b_1 \\ 0 & 1 & 1 & 0 & -1 & b_1 + b_2 \\ 0 & 0 & 0 & 1 & 1 & b_1 + b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & b_2 + b_3 + b_4 \end{array}\right)$$

Hence $A\mathbf{x} = \mathbf{b}$ is solvable if and only if $b_2 + b_3 + b_4 = 0$. In particular, the equation is not solvable for the given $\mathbf{b} = (1 \quad 1 \quad 1 \quad 1)^t$.

PROBLEM 3. (25 points) Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 2 & 4 & 1 \end{pmatrix}, \qquad \mathbf{b} := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{c} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

(i.) Find the QR-decomposition of A.

(ii.) Find the projection matrix onto $R(A)$.

(iii.) Find the projection matrix onto $N(A^t)$.

(iv.) Find the least square solution to $A\mathbf{x} = \mathbf{b}$

(v.) Find the minimal length solution to $A\mathbf{x} = \mathbf{c}$. Help: the QR-decomposition of A^t is

$$A^t = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{30} \\ 2/\sqrt{6} & 2/\sqrt{30} \\ 1/\sqrt{6} & -5/\sqrt{30} \end{pmatrix} \begin{pmatrix} \sqrt{6} & \frac{5}{3}\sqrt{6} & 11/\sqrt{6} \\ 0 & \frac{1}{3}\sqrt{30} & \frac{1}{6}\sqrt{30} \end{pmatrix}$$

SOLUTION:

(i) The first column of Q is just the first column of A normalized:

$$\mathbf{q}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

Since the second column of A is a multiple of the first, we do not get a new $\mathbf{q}$ vector from that. But you could also compute

$$\mathbf{w}_2 = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} - 6 \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where 6 is from the scalar product of $\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$.

Finally

$$\mathbf{w}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

where 1 is from the scalar product of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$. This vector is already normalized, so

$$\mathbf{q}_2 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

Hence

$$Q = \begin{pmatrix} 1/3 & 2/3 \\ 2/3 & -2/3 \\ 2/3 & 1/3 \end{pmatrix}$$

and from $R = Q^t A$

$$R = \begin{pmatrix} 3 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

You can check that

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 2/3 & -2/3 \\ 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 3 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) The projection onto $R(A)$ is

$$QQ^t = \frac{1}{9} \begin{pmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{pmatrix}$$

(iii) Since $N(A^t)$ is the orthogonal complement of $R(A)$, the projection is

$$I - QQ^t = \frac{1}{9} \begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{pmatrix}$$

(iv) The least square solution is obtained from $R\mathbf{x} = Q^t\mathbf{b}$, i.e.

$$\begin{pmatrix} 3 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Q^t\mathbf{b} = \begin{pmatrix} 5/3 \\ 1/3 \end{pmatrix}$$

Hence $x_3 = 1/3$ and x_1, x_2 can be anything satisfying

$$3x_1 + 6x_2 = \frac{4}{3}$$

For example $x_2 = 0$, $x_1 = 4/9$ will do.

(v). First find one solution to $A\mathbf{x} = \mathbf{c}$. You can run row reduction, but it is also easy to see just "by looking at it" that $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a solution. Then the minimal length solution is given by $\mathbf{x}^* := \widetilde{Q}\widetilde{Q}^t\mathbf{x}$, where $\widetilde{Q}$ is from the QR-decomposition of A^t, i.e.

$$\mathbf{x}^* = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{30} \\ 2/\sqrt{6} & 2/\sqrt{30} \\ 1/\sqrt{6} & -5/\sqrt{30} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{30} & -5/\sqrt{30} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix}$$

(Kitchen trick: it is faster if you do not multiply these two matrices together, but first you compute the second one acting on $\mathbf{x}$, then you let the first one acting on the resulting vector.)

PROBLEM 4. (25 points) Let

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} \right\}$$

and

$$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$$

be two bases in $\mathbf{R}^3$.

(i.) Write the vector $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ in the $\{\mathbf{v}\}$ basis.

(ii.) Find the change of basis matrix from the $\{\mathbf{v}\}$ basis to the $\{\mathbf{w}\}$ basis.

(iii.) Let $\mathbf{a} = 4\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$. Find its coordinates in the $\{\mathbf{w}\}$ basis.

(iv.) Let T be the reflection onto the plane spanned by $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Find its matrix in the standard basis.

(v.) Find the matrix of T in the $\{\mathbf{v}\}$ basis (i.e. ${}_{\mathbf{v}}M^T_{\mathbf{v}}$).

(vi.) What is $T(\mathbf{a})$ in the $\{\mathbf{v}\}$ basis? ($\mathbf{a}$ is given in (iii)).

(vii.) Let $\mathbf{b} = \mathbf{w}_1 + 2\mathbf{w}_2 + 3\mathbf{w}_3$. What is the matrix which immediately gives the coordinates of $T(\mathbf{b})$ in the $\{\mathbf{v}\}$ basis from the coordinate tuple $(1, 2, 3)$ of $\mathbf{b}$ in the $\{\mathbf{w}\}$ basis?

You might find the following relations useful:

$$\begin{pmatrix} 2 & 2 & 3 \\ -2 & 1 & -1 \\ -1 & -2 & -2 \end{pmatrix} \begin{pmatrix} -4 & -2 & -5 \\ -3 & -1 & -4 \\ 5 & 2 & 6 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix} \begin{pmatrix} 4 & -2 & -3 \\ 3 & -1 & -2 \\ 5 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

SOLUTION: As usual, let

$$V := \begin{pmatrix} 2 & 2 & 3 \\ -2 & 1 & -1 \\ -1 & -2 & -2 \end{pmatrix} \qquad W := \begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$$

be the matrices formed from these vectors. The inverse matrices were given as a help:

$$V^{-1} = \begin{pmatrix} -4 & -2 & -5 \\ -3 & -1 & -4 \\ 5 & 2 & 6 \end{pmatrix} \qquad W^{-1} = \begin{pmatrix} 4 & -2 & -3 \\ 3 & -1 & -2 \\ 5 & -2 & -3 \end{pmatrix}$$

(i)

$$V^{-1}\mathbf{u} = \begin{pmatrix} -4 & -2 & -5 \\ -3 & -1 & -4 \\ 5 & 2 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -39 \\ -29 \\ 46 \end{pmatrix}$$

i.e. $\mathbf{u} = -39\mathbf{v}_1 - 29\mathbf{v}_2 + 46\mathbf{v}_3$

(ii) This matrix is

$$W^{-1}V = \begin{pmatrix} 15 & 12 & 20 \\ 10 & 9 & 14 \\ 17 & 14 & 23 \end{pmatrix}$$

(iii)

$$W^{-1}V \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 56 \\ 36 \\ 63 \end{pmatrix}$$

hence $\mathbf{a} = 4\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = 56\mathbf{w}_1 + 36\mathbf{w}_2 + 63\mathbf{w}_3$.

(iv) This matrix has to flip the third coordinate (in the standard basis) and leave the other two invariant. Hence

$$M^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(v)

$${}_{\mathbf{v}}M^T_{\mathbf{v}} = V^{-1}M^TV = \begin{pmatrix} -4 & -2 & -5 \\ -3 & -1 & -4 \\ 5 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ -2 & 1 & -1 \\ -1 & -2 & -2 \end{pmatrix} = \begin{pmatrix} -9 & -20 & -20 \\ -8 & -15 & -16 \\ 12 & 24 & 25 \end{pmatrix}$$

(vi.)

$$V^{-1}M^TV \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 & -20 & -20 \\ -8 & -15 & -16 \\ 12 & 24 & 25 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -16 \\ -18 \\ 25 \end{pmatrix}$$

hence $T(\mathbf{a}) = -16\mathbf{v}_1 - 18\mathbf{v}_2 + 25\mathbf{v}_3$.

(vii.)

$${}_{\mathbf{v}}M^T_{\mathbf{w}} = V^{-1}M^TW = \begin{pmatrix} -4 & -2 & -5 \\ -3 & -1 & -4 \\ 5 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -16 & 8 \\ 0 & -11 & 6 \\ -1 & 18 & -9 \end{pmatrix}$$

PROBLEM 5. (10 points) Let $\mathcal{T}$ be the triangle with vertices $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$.

(i) Find the matrix of the linear transformation which takes the triangle into the following positions (prime denotes the image of the corresponding vertex):

$$A' = A, \quad B' = \Big(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\Big), \quad C' = (-\sqrt{2}, 0)$$

(ii) Find an affine transformation which takes the triangle into the given position. You can use only rotations and translations

$$A' = (-1, -1), \quad B' = (-1, -2), \quad C' = (0, -2)$$

SOLUTION

(i) This is a rotation with $3\pi/4$ followed by a reflection onto the x-axis, i.e. the matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

(ii) First solution: This is in fact a single rotation by $-\pi/2$ but around the point $P(-1, 0)$. Let $\mathbf{v} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ be the vector to this point. Hence first you translate the original picture by $-\mathbf{v}$ to get the rotation center P into the origin. The first step is $\mathbf{u} \to \mathbf{u} - \mathbf{v}$, or in coordinates

$$T_{-\mathbf{v}}\mathbf{u} = T_{-\mathbf{v}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ y \end{pmatrix}$$

The second step is the rotation, now around the origin, by $-\pi/2$. The matrix is $R_{-\pi/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and

$$R_{-\pi/2}(T_{-\mathbf{v}}\mathbf{u}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x+1 \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x-1 \end{pmatrix}$$

Finally you translate the picture back, i.e. apply $T_{\mathbf{v}}$:

$$T_{\mathbf{v}}\Big(R_{-\pi/2}(T_{-\mathbf{v}}\mathbf{u})\Big) = \begin{pmatrix} y \\ -x-1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} y-1 \\ -x-1 \end{pmatrix}$$

Second solution: If you don't see immediately the rotation center P, then you can try to rotate it first, so that the position of the rotated triangle be the same as the required one, and then translate. I.e. first apply a rotation $R_{-\pi/2}$, this gets the position right, but the vertex A is still in the origin. Hence after rotation you have to translate it to its place A', i.e. translate by $\mathbf{w} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. The result is

$$T_{\mathbf{w}}(R_{-\pi/2}(\mathbf{u})) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} y - 1 \\ -x - 1 \end{pmatrix}$$

PROBLEM 6. (8 points)

Decide whether the following statements are true (T) or false (F). If false, give a short argument.

a, If A is an $n \times k$ matrix with $n > k$, then its columns are linearly independent.

b, If A is an $n \times k$ matrix with $n < k$, then its columns are linearly dependent.

c, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions if the nullspace of A is nontrivial.

d, Suppose that A is invertible. Then A^t is also invertible and its inverse is the transpose of A^{-1}.

e, There exists such a 3×3 matrix A that $R(A) = N(A)$.

f, For any matrix A the nullspace $N(A)$ and the column space $R(A)$ are orthogonal complements of each other.

g, The set of orthogonal 3 by 3 matrices forms a vectorspace (with the usual matrix operations).

h, The projection onto the plane $x + 3y - 2z = 1$ in $\mathbf{R}^3$ is a linear transformation.

SOLUTION:

a, False. E.g. The matrix can have long identical columns.

b, True

c, False. You have to ensure that there exists a solution at all which would follow from $\mathbf{b} \in R(A)$ and has nothing to do with the nullspace

d, True

e, False. From the dimension formula the dimension of $R(A)$ and $N(A)$ must add up to 3, but if they are equal, then their dimensions d is also equal and $d + d = 3$ has no integer solution.

f, False. Recall that $N(A)$ and $R(A^t)$ are orthogonal complements. For general matrices $N(A)$ and $R(A)$ do not even lie in the same space.

g, False. The zero matrix is not orthogonal, but every vectorspace of matrices must contain the zero matrix.

h, False. This is a plane which does not go through the origin. In particular the reflection does not leave the origin invariant, but every linear transformation does.