March 29, 2000

2601C2 Calculus III for CS
Spring 2000

SOLUTION TO MIDTERM EXAM II.

PROBLEM 1. (20+10 points) Consider the following symmetric matrix

$$A := \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

(i) Find the spectral decomposition of A. (Advice: the eigenvalues are "nice" numbers. If you got "ugly" numbers, you made a mistake which would overcomplicate the rest.)

(ii) Find the solution to the following system of differential equations

$$\begin{array}{rcrcrcr} x_1'(t) & = & 2x_1(t) & - & x_2(t) & - & x_3(t) \\ x_2'(t) & = & -x_1(t) & + & 2x_2(t) & - & x_3(t) \\ x_3'(t) & = & -x_1(t) & - & x_2(t) & + & 2x_3(t) \end{array}$$

with initial conditions $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 0$.

SOLUTION: (i) First we find the eigenvalues, i.e. we start with the characteristic polynomial:

$$det \begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)^3 + 2 \cdot (-1)^3 - 3(-1)^2(2-\lambda) = -\lambda^3 + 6\lambda^2 - 9\lambda$$

Clearly $\lambda_1 = 0$, then we have to solve the remaining quadratic equation $\lambda^2 - 6\lambda + 9 = 0$. Either by the root formula, or just by noticing that it is a complete square $(\lambda - 3)^2$, you find that $\lambda_2 = \lambda_3 = 3$.

Now we look for eigenvectors. The eigenvector $\mathbf{u}_1$ belonging to $\lambda_1 = 0$ is easy to find, its coordinates satisfy

$$\begin{array}{rcrrcl} 2x & - & y & -z & = & 0 \\ -x & + & 2y & -z & = & 0 \\ -x & - & y & +2z & = & 0 \end{array}$$

and either you solve it honestly by Gaussian elimination, or notice that $x = y = z$ is a solution. Hence the eigenvector is

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and we can easily normalize it

$$\mathbf{q}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The eigenvalue $\lambda_2 = \lambda_3 = 3$ is double. The equation for eigenvectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with eigenvalue 3 is

$$\begin{array}{rcrcrcl} -x & - & y & - & z & = & 0 \\ -x & - & y & - & z & = & 0 \\ -x & - & y & - & z & = & 0 \end{array}$$

i.e. it is really just one equation. If you absolutely honestly run the elimination, you should notice that there are two free variables (y, z), hence the solution space is two dimensional. Notice that it MUST BE like that, since the original matrix was symmetric, and that there are as many eigenvectors as the multiplicity predicts

One way to actually find these vectors is that first we find two linearly independent eigenvectors, then we orthogonalize them by Gram-Schmidt. For example

$$\mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

are clearly solutions to the above equation (choosing one free variable nonzero, the other one zero and then repeat it in the other way around)

By Gram-Schmidt we get

$$\mathbf{q}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

and

$$\mathbf{w}_3 = \mathbf{u}_3 - (\mathbf{u}_3^t \cdot \mathbf{q}_2)\mathbf{q}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$$

hence

$$\mathbf{q}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Therefore the (orthogonal) matrix of eigenvectors is

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$$

Hence the spectral decomposition (or diagonalization) of A is

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} = QDQ^t$$

You can also write it as

$$A = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \mathbf{u}_i^t = 0 \cdot \mathbf{u}_1 \mathbf{u}_1^t + 3\mathbf{u}_2 \mathbf{u}_2^t + 3\mathbf{u}_3 \mathbf{u}_3^t$$

$$= 0 \cdot \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} + 3 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 1/6 & 1/6 & -1/3 \\ 1/6 & 1/6 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

You can combine the second two into one matrix, since the eigenvalue is common. Hence the spectral decomposition is

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} + 3 \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

(it looks stupid to keep the first term, since this is zero, but when we take functions of A, this may become nonzero again).

(ii) The solution to the system of differential equation is

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = e^{At}\mathbf{x}(0) = e^{At} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

To compute e^{At}, it's the best to use the second form of the spectral decomposition above:

$$e^{At} = \sum_{i=1}^{3} e^{\lambda_i t}\mathbf{u}_i\mathbf{u}_i^t = e^{-0\cdot t}\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} + e^{3t}\begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

Of course $e^{-0\cdot t} = 1$. Now we compute how this acts on $\mathbf{x}(0)$:

$$\mathbf{x}(t) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + e^{3t}\begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 + e^{3t} \\ 1 - e^{3t} \end{pmatrix}$$

PROBLEM 2. (25 points) Find the QR-decomposition of the matrix

$$B := \begin{pmatrix} 3 & 2 & 5 \\ 0 & 1 & 2 \\ 4 & 1 & 0 \end{pmatrix}$$

using the Householder method.

SOLUTION: The norm of the first column is $\sqrt{3^2+4^2} = 5$, hence the first Householder vector is

$$\mathbf{u} = \begin{pmatrix} 3+5 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix}$$

Since only its direction matters, we can use the simpler

$$\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

vector instead (divide by 4). The Householder matrix is

$$H_1 = I - \frac{2\mathbf{u}\mathbf{u}^t}{\|\mathbf{u}\|^2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{pmatrix}$$

Hence

$$H_1 B = \begin{pmatrix} -3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{pmatrix}\begin{pmatrix} 3 & 2 & 5 \\ 0 & 1 & 2 \\ 4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -5 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & -1 & -4 \end{pmatrix}$$

The next (reduced) column vector to eliminate is $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, with norm $\|\mathbf{x}\| = \sqrt{2}$. Hence the new normal vector is

$$\mathbf{u} = \begin{pmatrix} 1+\sqrt{2} \\ -1 \end{pmatrix}$$

with norm square $\|\mathbf{u}\|^2 = 4 + 2\sqrt{2}$. Hence the second (still reduced) Householder matrix is

$$\begin{aligned} \widetilde{H}_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{4+2\sqrt{2}}\begin{pmatrix} 1+\sqrt{2} \\ -1 \end{pmatrix}\begin{pmatrix} 1+\sqrt{2} & -1 \end{pmatrix} \\ &= \frac{1}{4+2\sqrt{2}}\begin{pmatrix} (4+2\sqrt{2}) - 2(3+2\sqrt{2}) & 2(1+\sqrt{2}) \\ 2(1+\sqrt{2}) & (4+2\sqrt{2}) - 2 \end{pmatrix} \\ &= \frac{1}{4+2\sqrt{2}}\begin{pmatrix} -2(1+\sqrt{2}) & 2(1+\sqrt{2}) \\ 2(1+\sqrt{2}) & 2(1+\sqrt{2}) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

If you did not like square roots, you could compute with decimals: The norm of $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is 1.41, hence

$$\mathbf{u} = \begin{pmatrix} 2.41 \\ -1 \end{pmatrix}$$

with norm square $\|\mathbf{u}\|^2 = 2.41^2 + 1 = 6.8$, so

$$\widetilde{H}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{6.8} \begin{pmatrix} 2.41 \\ -1 \end{pmatrix} \begin{pmatrix} 2.41 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1.708 & -0.708 \\ -0.708 & 0.293 \end{pmatrix} = \begin{pmatrix} -0.708 & 0.708 \\ 0.708 & 0.707 \end{pmatrix}$$

In any case, the second Householder matrix is

$$H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.707 & 0.707 \\ 0 & 0.707 & 0.707 \end{pmatrix}$$

and

$$H_2(H_1B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -5 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & -1 & -4 \end{pmatrix} = \begin{pmatrix} -5 & -3 & -2 \\ 0 & -\sqrt{2} & -3\sqrt{2} \\ 0 & 0 & -\sqrt{2} \end{pmatrix} =: R$$

and this is the R matrix of the QR-decomposition of B. Finally

$$Q = H_1H_2 = \begin{pmatrix} -3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -3/5 & -2\sqrt{2}/5 & -2\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -4/5 & 3\sqrt{2}/10 & 3\sqrt{2}/10 \end{pmatrix}$$

If you used decimals, you should have obtained

$$R := \begin{pmatrix} -5 & -2 & -3 \\ 0 & -1.414 & -4.242 \\ 0 & 0 & -1.414 \end{pmatrix} \qquad Q = \begin{pmatrix} -0.6 & -0.565 & -0.565 \\ 0 & -0.707 & 0.707 \\ -0.8 & 0.424 & 0.424 \end{pmatrix}$$

PROBLEM 3. (15+5 points) (i) Find **both** eigenvalues of

$$C := \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix}$$

by the power method, up to two digit precision (you are not allowed to compute the characteristic polynomial). You may find it useful that $C^{-1} = \begin{pmatrix} 1/2 & -1 \\ -1 & 5/2 \end{pmatrix}$

(ii) Your boss asks you to solve the system of equations

$$\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \mathbf{x} = \mathbf{b}$$

with a relative error smaller than 10^{-3} in the solution $\mathbf{x}$, where $\mathbf{b}$ is a measured data. Roughly, how precise data $\mathbf{b}$ do you have to ask for? (I.e., at most how large can the relative error of $\mathbf{b}$ be so that the solution $\mathbf{x}$ have a relative error smaller than 10^{-3}?) Hint: you will need the result of part (i).

SOLUTION: (i) Start essentially with any vector, for example $\mathbf{u}^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and run the iteration $\mathbf{u}^{(n)} = C\mathbf{u}^{(n-1)}$. You get

$$\mathbf{u}^{(1)} = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 6 \end{pmatrix}$$

and we check (for example) the ratio of the first entries: $\frac{14}{1} = 14$. Then

$$\mathbf{u}^{(2)} = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 14 \\ 6 \end{pmatrix} = \begin{pmatrix} 164 \\ 68 \end{pmatrix}$$

and the ratio of the first entries is $\frac{164}{14} = 11.71$. Then

$$\mathbf{u}^{(3)} = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 164 \\ 68 \end{pmatrix} = \begin{pmatrix} 1912 \\ 742 \end{pmatrix}$$

and the ratio of the first entries is $\frac{1912}{164} = 11.65$, so the first two digits are stable, $\lambda_1 \approx 11$. This is the bigger (in modulus) eigenvalue of C.

The smaller eigenvalue of C is the inverse of the bigger eigenvalue of C^{-1}, i.e. we run the same iteration for C^{-1}:

$$\mathbf{u}^{(1)} = \begin{pmatrix} 1/2 & -1 \\ -1 & 5/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 1.5 \end{pmatrix}$$

and we check the ratio of the first entries: $\frac{-0.5}{1} = -0.5$. Then

$$\mathbf{u}^{(2)} = \begin{pmatrix} 1/2 & -1 \\ -1 & 5/2 \end{pmatrix} \begin{pmatrix} -0.5 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -1.75 \\ 4.25 \end{pmatrix}$$

and the ratio of the first entries: $\frac{-1.75}{-0.5} = 3.5$. Then

$$\mathbf{u}^{(3)} = \begin{pmatrix} 1/2 & -1 \\ -1 & 5/2 \end{pmatrix} \begin{pmatrix} -1.75 \\ 4.25 \end{pmatrix} = \begin{pmatrix} -5.125 \\ 12.37 \end{pmatrix}$$

and the ratio of the first entries: $\frac{-5.125}{-1.75} = 2.928$. Then

$$\mathbf{u}^{(4)} = \begin{pmatrix} 1/2 & -1 \\ -1 & 5/2 \end{pmatrix} \begin{pmatrix} -5.12 \\ 12.3 \end{pmatrix} = \begin{pmatrix} -14.9 \\ 36 \end{pmatrix}$$

and the ratio of the first entries: $\frac{-14.9}{-5.12} = 2.9$, so two digits are stable, the bigger eigenvalue of C^{-1} is 2.9, hence the smaller eigenvalue of C is $1/2.9 = 0.34$.

(ii) Here we have to compute the condition number of $D := \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$. Recall that this is the square root of the ratio of the biggest and smallest eigenvalues of $DD^t = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix}$, which we just computed. Hence

$$cond(D) = \sqrt{\frac{\lambda_{max}(DD^t)}{\lambda_{min}(DD^t)}} = \sqrt{\frac{11}{0.34}} = 5.68$$

Hence the relative error in $\mathbf{x}$ is at most 5.68 times the relative error in $\mathbf{b}$, so if you want a relative error not bigger than 10^{-3} in $\mathbf{x}$, you should ask for a relative error not bigger than $10^{-3}/5.68 \approx 10^{-4}$ in the measured data (notice that really the order of magnitude matters, i.e. whether it is 10^{-3}, 10^{-4} or 10^{-8}, for example).

PROBLEM 4. (10+7 points) Let

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(i) Does the Gauss-Seidel iteration converge for $A\mathbf{x} = \mathbf{b}$? If so, find an approximate solution to $A\mathbf{x} = \mathbf{b}$ with the Gauss-Seidel iteration starting from the zero vector. Run two steps of the iteration.

(ii) Estimate roughly how many iteration steps would be needed to find the solution up to thirty digit precision using Gauss-Seidel?

SOLUTION:

(i) Gauss-Seidel converges, since the matrix A is clearly diagonally dominant. The iteration formula in our case is

$$x_1^{(n)} = \frac{1}{3}\Big[1 - x_2^{(n-1)}\Big]$$

$$x_2^{(n)} = \frac{1}{5}\Big[2 - 2x_1^{(n)}\Big]$$

So from $\mathbf{x}^{(0)} = \mathbf{0}$ we have

$$x_1^{(1)} = \frac{1}{3} = 0.333$$

$$x_2^{(1)} = \frac{1}{5}\Big[2 - 2 \cdot \frac{1}{3}\Big] = \frac{4}{15} = 0.266$$

So the first vector is $\mathbf{x}^{(1)} = \begin{pmatrix} 0.333 \\ 0.266 \end{pmatrix}$. The next one:

$$x_1^{(2)} = \frac{1}{3}\Big[1 - \frac{4}{15}\Big] = \frac{11}{45} = 0.244$$

$$x_2^{(2)} = \frac{1}{5}[2 - 2 \cdot \frac{11}{45}] = \frac{68}{225} = 0.302$$

The second vector is $\mathbf{x}^{(2)} = \begin{pmatrix} 0.244 \\ 0.302 \end{pmatrix}$.

(The true solution, by the way, is $\mathbf{x} = \begin{pmatrix} 3/13 \\ 4/13 \end{pmatrix} = \begin{pmatrix} 0.23 \\ 0.307 \end{pmatrix}$)

(ii) The speed of convergence of Gauss-Seidel is given by the biggest (in modulus) eigenvalue of the matrix

$$(D+L)^{-1}U = \begin{pmatrix} 1/3 & 0 \\ -2/15 & 1/5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/3 \\ 0 & -2/15 \end{pmatrix}$$

This matrix has two eigenvalues: 0 and $-2/15$, so the speed of convergence is given by $|-2/15| = 0.13$. We need at least n steps where

$$0.13^n \leq 10^{-30}$$

i.e.

$$n \geq \frac{30}{-\log_{10} 0.13} = 33.85$$

so we need 34 or 35 steps.

PROBLEM 5. (8 points) Decide whether the following statements are true (T) or false (F). If false, give a short argument.

a, Every regular square matrix is diagonalizable

False. The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is regular, but not diagonalizable.

b, Every matrix has a singular value decomposition. TRUE

c, Every symmetric matrix is diagonalizable. TRUE

d, The condition number of an orthogonal matrix is always 1. TRUE

e, Suppose that A is an invertible matrix. Then $\|A^{-1}\| = \frac{1}{\|A\|}$.

False. E.g. the norm of $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is 2, but $\|A^{-1}\| = 1$.

f, The most stable method to solve $A\mathbf{x} = \mathbf{b}$ (among the ones we learned) is via the LU factorization.

False. LU can be very unstable; QR via Householder or Givens are the stablest.

g, The fixed point method to solve $x = f(x)$ is always convergent if you start from a point lying in a small enough interval around the solution.

False. You also need $|f'(\alpha)| < 1$, where α is the fixed point.

h, The QR-iteration diagonalizes any square matrix.

False. QR-iteration in general gives only the eigenvalues (even those only if they have distinct modulus). For symmetric matrices is gives the full diagonalization. By the way, this statement cannot be true by part a, since not every square matrix is diagonalizable.