## Chapter Fourteen

## One Dimension Again

### 14.1 Scalar Line Integrals

Now we again consider the idea of the integral in one dimension. When we were introduced to the integral back in elementary school, we considered only functions defined on nice subsets of the real line. The notion of an integral of a function $f: \boldsymbol{D} \rightarrow \boldsymbol{R}$ in which $\boldsymbol{D}$ is a nice one dimensional set, but is not a subset of the reals is our next object of study. To get some idea of why one might care about such a thing, consider the simple problem of finding the mass of a piece of wire having the shape of an arc of a space curve $\boldsymbol{C}$ and having a given density $\rho(\boldsymbol{r})$. How might we approach such a problem? Simple enough! We subdivide, or partition, the curve with a finite set of points, say $\left\{\boldsymbol{r}_{0}, \boldsymbol{r}_{1}, \ldots, \boldsymbol{r}_{n}\right\}$. On the subarc joining $\boldsymbol{r}_{i-1}$ to $\boldsymbol{r}_{i}$, we choose a point, say $\boldsymbol{r}_{i}^{*}$, and evaluate the function $\rho\left(\boldsymbol{r}_{i}^{*}\right)$. Now we multiply this times the length of the line segment joining the points $\boldsymbol{r}_{i-1}$ and $\boldsymbol{r}_{i}$ for an approximation to the mass of this arc of our curve. Then sum these to obtain an approximation for the total mass:

$$
S=\sum_{i=1}^{n} \rho\left(\boldsymbol{r}_{i}^{*}\right)\left|\boldsymbol{r}_{i}-\boldsymbol{r}_{i-1}\right|
$$

Then we all believe that the "limit" of these sums as we choose finer and finer partitions of the curve should be the actual, honest-to-goodness mass of the wire.

Let's abstract the essence of the discussion. Suppose $f: \boldsymbol{C} \rightarrow \boldsymbol{R}$ is a function whose domain $\boldsymbol{C}$ is a curve (in $\boldsymbol{R}^{2}$ or $\boldsymbol{R}^{3}$, or wherever). We subdivide the curve as in the preceding discussion and choose a point $\boldsymbol{r}_{i}^{*}$ on the subarc joining $\boldsymbol{r}_{i-1}$ to $\boldsymbol{r}_{\boldsymbol{i}}$. The sum

$$
S=\sum_{i=1}^{n} f\left(\boldsymbol{r}_{i}^{*}\right)\left|\boldsymbol{r}_{i}-\boldsymbol{r}_{i-1}\right|
$$

again is called a Riemann sum. If there is a number $L$ such that all Riemann sums are arbitrarily close to $L$ for sufficiently fine partitions, then we say $f$ is integrable on $\boldsymbol{C}$, and the number $L$ is called the integral of $\boldsymbol{f}$ on $\boldsymbol{C}$ and is denoted $\int_{\boldsymbol{C}} f(\boldsymbol{r}) d \boldsymbol{r}$. This integral is also frequently referred to as a line integral.


This is wonderful, but how do find such an integral? It is remarkably simple and easy. Suppose we have a vector description of the curve $\boldsymbol{C}$; say $\boldsymbol{r}(t)$, for $a \leq t \leq b$. We partition the curve by partitioning the interval $[a, b]$ : If $\left\{a=t_{0}, t_{1}, \ldots, t_{n}=b\right\}$ is a partition of the interval, then the points $\left\{\boldsymbol{r}\left(t_{0}\right), \boldsymbol{r}\left(t_{1}\right), \ldots, \boldsymbol{r}\left(t_{n}\right)\right\}$ partition the curve $\boldsymbol{C}$. We obtain the point $\boldsymbol{r}_{i}^{*}$ on the subarc joining $\boldsymbol{r}\left(t_{i-1}\right)$ to $\boldsymbol{r}\left(t_{i}\right)$ by choosing $t_{i}^{*} \in\left[t_{i-1}, t_{i}\right]$ and letting $\boldsymbol{r}_{i}^{*}=\boldsymbol{r}\left(t_{i}^{*}\right)$. Our Riemann sum now looks like

$$
S=\sum_{i=1}^{n} f\left(\boldsymbol{r}\left(t_{i}^{*}\right)\left|\boldsymbol{r}\left(t_{i}\right)-\boldsymbol{r}\left(t_{i-1}\right)\right| .\right.
$$

Next, multiply the terms on the right by one, but one disguised as $\frac{\Delta t_{i}}{\Delta t_{i}}$, where, of course, $\Delta t_{i}=t_{i}-t_{i-1}$. Then we see

$$
S=\sum_{i=1}^{n} f\left(\boldsymbol{r}\left(t_{i}^{*}\right)\left|\frac{\boldsymbol{r}\left(t_{i}\right)-\boldsymbol{r}\left(t_{i-1}\right)}{\Delta t_{i}}\right| \Delta t_{i} .\right.
$$

We know that $\lim _{\Delta t \rightarrow 0}\left|\frac{\boldsymbol{r}\left(t_{i}\right)-\boldsymbol{r}\left(t_{i-1}\right)}{\Delta t_{i}}\right|=\left|\frac{d \boldsymbol{r}}{d t}\right|$, and so it is not hard to convince oneself that the "limiting" value of the Riemann sums is

$$
\int_{a}^{b} f(\boldsymbol{r}(t))\left|\frac{d \boldsymbol{r}(t)}{d t}\right| d t
$$

We have thus turned the problem into one we know how to solve-a plain old everyday elementary calculus integral. Hence,

$$
\int_{C} f(\boldsymbol{r}) d \boldsymbol{r}=\int_{a}^{b} f(\boldsymbol{r}(t))\left|\frac{d \boldsymbol{r}(t)}{d t}\right| d t
$$

## Example

Suppose we have a wire in the shape of a quarter circle of radius 2 , and the density of the wire is given by $\rho(x, y)=y$. What is the mass of the wire? Well, we know the mass is simply the integral $\int_{C} y d \boldsymbol{r}$, where $\boldsymbol{C}$ is the quarter circle:


A vector description of the curve is $\boldsymbol{r}(t)=2 \cos t \boldsymbol{i}+2 \sin t \boldsymbol{j}$, for $0 \leq t \leq \frac{\pi}{4}$. Thus we have $\left|\frac{d r}{d t}\right|=|-2 \sin t i+2 \cos t j|=2$, and the integral becomes simply

$$
\int_{C} y d \boldsymbol{r}=\int_{0}^{\pi / 4} 4 \sin t d t=4
$$

Let's see what happens if we use a different vector description of the curve, say $\boldsymbol{r}(t)=t \boldsymbol{i}+\sqrt{4-t^{2}} \boldsymbol{j}$ for $0 \leq t \leq 2$. We have $\left|\frac{d \boldsymbol{r}}{d t}\right|=\left|\boldsymbol{i}-\frac{t}{\sqrt{4-t^{2}}} \boldsymbol{j}\right|=\frac{2}{\sqrt{4-t^{2}}}$. Hence

$$
\int_{C} y d r=\int_{0}^{2} \sqrt{4-t^{2}}\left(\frac{2}{\sqrt{4-t^{2}}}\right) d t=\int_{0}^{2} 2 d t=4
$$

We get, as we must, the same answer.

## Exercises

1. Evaluate the integral $\int_{\boldsymbol{C}}(x-y+z+2) d \boldsymbol{r}$, where $\boldsymbol{C}$ is the curve $\boldsymbol{r}(t)=\boldsymbol{i}+(1-t) \boldsymbol{j}+\boldsymbol{k}$, $0 \leq t \leq 1$.
2. Evaluate the integral $\int_{\boldsymbol{C}} \sqrt{x^{2}+y^{2}} d \boldsymbol{r}$, where $\boldsymbol{C}$ is the curve $\boldsymbol{r}(t)=4 \cos t \boldsymbol{i}+4 \sin t \boldsymbol{j}+3 t \boldsymbol{k}$, $-2 \pi \leq t \leq 2 \pi$.
3. Find the centroid of a semicircle of radius $a$.
4. Find the mass of a wire having the shape of the curve $\boldsymbol{r}(t)=\left(t^{2}-1\right) \boldsymbol{j}+2 t \boldsymbol{k}, 0 \leq t \leq 1$ if the density is $\rho(t)=\frac{3}{2} t$.
5. Find the center of mass of a wire having the shape of the curve

$$
\boldsymbol{r}(t)=t \boldsymbol{i}+\frac{2 \sqrt{2}}{3} t^{3 / 2} \boldsymbol{j}+\frac{t^{2}}{2} \boldsymbol{k}, 0 \leq t \leq 2
$$

if the density is $\rho(t)=\frac{1}{t+1}$.
6. What is $\int_{C} d r$ ?

### 14.2 Vector Line Integrals

Now we are introduce something perhaps a little different from what we have seen to now-integrals with vector valued integrands. Specifically, suppose $\boldsymbol{C}$ is a space curve and $\boldsymbol{f}: \boldsymbol{C} \rightarrow \boldsymbol{R}^{3}$ is a function from $\boldsymbol{C}$ into the Euclidean space $\boldsymbol{R}^{3}$. We are going to define an integral $\int_{C} f(\boldsymbol{r}) \cdot d \boldsymbol{r}$. Why should we care about such a thing? Again, let's think about a physical model. You learned in fifth grade physics that the work done by a force $F$ acting through a distance $d$ is simply the product $F d$. The force $F$ and the displacement $d$ are, of course, really vectors, and we saw earlier in life that the "product" of the two is actually
the scalar, or dot, product of the two vectors. Now, in general, neither of these quantities will be constant, and we will have a variable force $\boldsymbol{F}(\boldsymbol{r})$ acting along a curve $\boldsymbol{C}$ in space. How do we compute the work done in this situation? Let's see. Once more, we partition the curve by choosing a sequence of points $\left\{\boldsymbol{r}_{0}, \boldsymbol{r}_{1}, \ldots, \boldsymbol{r}_{n}\right\}$ on the curve, with $\boldsymbol{r}_{0}$ being the initial point and $\boldsymbol{r}_{n}$ being the final point. Now, of course, there is an orientation, or direction, specified on the curve. One may think of specifying an orientation by simply putting an arrow on the curve-it thus makes sense to speak of the initial point and the terminal point of the curve. Exactly as in the scalar integrand case, we choose a point $\boldsymbol{r}_{i}^{*}$ on the subarc joining $\boldsymbol{r}_{i-1}$ to $\boldsymbol{r}_{i}$, and evaluate $\boldsymbol{F}\left(\boldsymbol{r}_{i}^{*}\right)$. Now then, the work done in going from $\boldsymbol{r}_{i-1}$ to $\boldsymbol{r}_{i}$ is approximately the scalar product $\boldsymbol{F}\left(\boldsymbol{r}_{i}^{*}\right) \cdot\left(\boldsymbol{r}_{\boldsymbol{i}}-\boldsymbol{r}_{i-1}\right)$. Add all these up for an approximation to the total work done:

$$
S=\sum_{i=1}^{n} \boldsymbol{F}\left(\boldsymbol{r}_{i}^{*}\right) \cdot\left(\boldsymbol{r}_{i}-\boldsymbol{r}_{i-1}\right) .
$$

The course should be obvious now; we take finer and finer partitions, and the limiting value of the sums is the integral

$$
\int_{C} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r} .
$$

This integral too is called a line integral. To prevent confusion, we sometimes speak of scalar line integrals and vector line integrals. How to find such a vector integral should be clear from the discussion of scalar line integrals. We let $\boldsymbol{r}(t), a \leq t \leq b$, be a vector description of $\boldsymbol{C}$. (Here $\boldsymbol{r}(a)$ is the initial point and $\boldsymbol{r}(b)$ is the terminal point.) The discussion proceeds almost exactly as it did in the previous section and we get

$$
\int_{\boldsymbol{C}} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}=\int_{a}^{b} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d \boldsymbol{r}}{d t} d t
$$

## Example

Find $\int_{\boldsymbol{C}}\left[\left(x y+z^{2}\right) \boldsymbol{i}+(x+z) \boldsymbol{j}+2 y z \boldsymbol{k}\right] \cdot d \boldsymbol{r}$, where $\boldsymbol{C}$ is the straight line from the origin to the point $(1,2,3)$. The line $\boldsymbol{C}$ has a vector description $\boldsymbol{r}(t)=\boldsymbol{t}+2 \boldsymbol{t} \boldsymbol{j}+3 \boldsymbol{t k}$. Thus, $\frac{d \boldsymbol{r}}{d t}=\boldsymbol{i}+2 \boldsymbol{j}+3 \boldsymbol{k}$, and so

$$
\begin{aligned}
\int_{\boldsymbol{C}}\left[\left(x y+z^{2}\right) \boldsymbol{i}+(x+z) \boldsymbol{j}+2 y z \boldsymbol{k}\right] \cdot d \boldsymbol{r} & =\int_{0}^{1}\left[\left(2 t^{2}+9 t^{2}\right) \boldsymbol{i}+(t+3 t) \boldsymbol{j}+12 t^{2} \boldsymbol{k}\right] \cdot(\boldsymbol{i}+2 \boldsymbol{j}+3 \boldsymbol{k}) d t \\
& =\int_{0}^{1}\left(2 t^{2}+8 t+36 t^{2}\right) d t=\int_{0}^{1}\left(38 t^{2}+8 t\right) d t \\
& =\frac{38}{3} t^{3}+\left.4 t^{2}\right|_{0} ^{1}=\frac{50}{3}
\end{aligned}
$$

Nothing to it.

## Another Example

Now let's integrate the same function from $(0,0,0)$ t0 $(1,2,3)$, but this time along the path $P$ in the picture:


Here the path $P$ is the union of the three nice curves, $P_{1}, P_{2}$, and $P_{3}$, so our integral is the sum of three integrals:

$$
\int_{P} \boldsymbol{F}(x, y, x) \cdot d \boldsymbol{r}=\int_{P_{1}} \boldsymbol{F}(x, y, x) \cdot d \boldsymbol{r}+\int_{P_{2}} \boldsymbol{F}(x, y, x) \cdot d \boldsymbol{r}+\int_{P_{3}} \boldsymbol{F}(x, y, x) \cdot d \boldsymbol{r},
$$

where

$$
\boldsymbol{F}(x, y, z)=\left(x y+z^{2}\right) \boldsymbol{i}+(x+z) \boldsymbol{j}+2 y \boldsymbol{k} .
$$

A vector description of $P_{1}$ is simply $\boldsymbol{r}(t)=\boldsymbol{t i}, 0 \leq t \leq 1$. Thus

$$
\int_{P_{1}} \boldsymbol{F}(x, y, z) \cdot d \boldsymbol{r}=\int_{0}^{1} \boldsymbol{F}(t, 0,0) \cdot \boldsymbol{i} d t=\int_{0}^{1} \boldsymbol{j} \cdot \boldsymbol{i} d t=0 .
$$

For $P_{2}$, we have $\boldsymbol{r}(t)=\boldsymbol{i}+\boldsymbol{j}, 0 \leq \mathrm{t} \leq 2$. This gives us

$$
\int_{P_{2}} \boldsymbol{F}(x, y, z) \cdot d \boldsymbol{r}=\int_{0}^{2} \boldsymbol{F}(1, t, 0) \cdot \boldsymbol{j} d t=\int_{0}^{2}(t \boldsymbol{i}+\boldsymbol{j}) \cdot \boldsymbol{j} d t=\int_{0}^{2} d t=2
$$

Finally, for $P_{3}$, there is $\boldsymbol{r}(t)=\boldsymbol{i}+2 \boldsymbol{j}+\boldsymbol{k}, 0 \leq t \leq 3$; and so

$$
\begin{gathered}
\int_{P_{3}} \boldsymbol{F}(x, y, z) \cdot d \boldsymbol{r}=\int_{0}^{3} \boldsymbol{F}(1,2, t) \cdot \boldsymbol{k} d t=\int_{0}^{3}\left[\left(2+t^{2}\right) \boldsymbol{i}+(1+t) \boldsymbol{j}+4 t \boldsymbol{k}\right] \cdot \boldsymbol{k} d t \\
=\int_{0}^{3} 4 t d t=18
\end{gathered}
$$

At last, we have then $\int_{P} \boldsymbol{F}(x, y, z) \cdot d \boldsymbol{r}=0+2+18=20$.

## Exercises

7. Evaluate $\int_{C}\left[x y \boldsymbol{i}+x^{2} \boldsymbol{j}\right] \cdot d \boldsymbol{r}$, where $C$ is the arc of the curve $y=x^{2}$ from $(0,0)$ to $(1,1)$.
8. Evaluate $\int_{C}(\cos x \boldsymbol{i}-y \boldsymbol{j}) \cdot d \boldsymbol{r}$ where $C$ the part of the curve $y=\sin x$ from ( 0,0 ) to $(\pi, 0)$.
9. Evaluate the line integral of $\boldsymbol{F}(x, y, z)=x y \boldsymbol{i}+(x y+y z) \boldsymbol{j}+z^{2} \boldsymbol{k}$ from $(0,0,0)$ to $(-1,1,2)$ along the line segment joining these two points.
10. Evaluate the line integral of $\boldsymbol{F}(x, y, z)=(x-z) \boldsymbol{i}+(y-z) \boldsymbol{j}-(x+y) \boldsymbol{k}$ along the polygonal path from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$.
11. Integrate $\boldsymbol{F}(x, y)=\frac{1}{x^{2}+y^{2}}(-y \boldsymbol{i}+x \boldsymbol{j})$ one time around the circle $x^{2}+y^{2}=a^{2}$ in the counterclockwise direction.

### 14.3 Path Independence

Suppose we evaluate the vector line integral $\int_{C} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}$, where $C$ is a curve from the point $\boldsymbol{p}$ to the point $\boldsymbol{q}$. Let $\boldsymbol{r}(t), a \leq t \leq b$, be a vector description of $C$. Then, of course, we have $\boldsymbol{r}(a)=\boldsymbol{p}$ and $\boldsymbol{r}(b)=\boldsymbol{q}$. As we have already seen,

$$
\int_{c} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}=\int_{a}^{b} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d \boldsymbol{r}}{d t} d t .
$$

Now let us make the very special assumption that there exists a real-valued (or scalar) function $g: \boldsymbol{R}^{3} \rightarrow \boldsymbol{R}$ such that the derivative, or gradient, of $g$ is the integrand $\boldsymbol{F}$ :

$$
\nabla g=\boldsymbol{F} .
$$

Next let's use the Chain Rule to compute the derivative of the composition $h(t)=g(\boldsymbol{r}(t)):$

$$
h^{\prime}(t)=\nabla g \cdot \frac{d \boldsymbol{r}}{d t}=\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d \boldsymbol{r}}{d t} .
$$

This is, mirabile dictu, precisely the integrand in our line integral:

$$
\int_{c} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}=\int_{a}^{b} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d \boldsymbol{r}}{d t} d t=\int_{a}^{b} h^{\prime}(t) d t=h(b)-h(a)=g(\boldsymbol{p})-g(\boldsymbol{q}) .
$$

This is a very exciting result and calls for some meditation. Note that the curve $C$ has completely disappeared from the answer. The value of the integral depends only on the values of the function $g$ at the endpoints; the path from $\boldsymbol{p}$ to $\boldsymbol{q}$ does not affect the answer. The line integral is path independent. The result is esthetically pleasing and is clearly the lineal descendant of the fundamental theorem of calculus we learned so many years ago.

A moment's reflection on the examples we have seen should convince us that a lot of integrals are not path independent, thus many very nice functions $\boldsymbol{F}$ (or vector fields) are not the gradient of any function. A function $\boldsymbol{F}$ that is the gradient of a function $g$ is said to be conservative and the function $g$ is said to be a potential function for $\boldsymbol{F}$.

Let's suppose the domain $\boldsymbol{D}$ of the function $\boldsymbol{F}: \boldsymbol{D} \rightarrow \boldsymbol{R}^{3}$ is open and connected (Thus any two points in $\boldsymbol{D}$ may be joined by a nice path.) We have just seen that if there exists a function $g: \boldsymbol{D} \rightarrow \boldsymbol{R}$ such that $\boldsymbol{F}=\nabla g$, then the integral of $\boldsymbol{F}$ between any two points of $\boldsymbol{D}$ does not depend on the path between the two points. It turns out, as we shall see, that the converse of this is true. Specifically, if every integral of $\boldsymbol{F}$ in $\boldsymbol{D}$ is path independent, then there is a function $g$ such that $\boldsymbol{F}=\nabla g$. Let's see why this is so.

Choose a point $\boldsymbol{p}=\left(x_{0}, y_{0}, z_{0}\right) \in \boldsymbol{D}$. Now define $g(\boldsymbol{s})=g(x, y, z)$ to be the integral from $\boldsymbol{p}$ to $\boldsymbol{s}$ along any curve joining these points. We are assuming path independence of the integral, so it matters not what curve we choose. Okay, now we compute the partial derivative $\frac{\partial g}{\partial x}$. The domain $\boldsymbol{D}$ is open and hence includes an open ball centered at $\boldsymbol{s}=(x, y, z) \in \boldsymbol{D}$. Choose a point $\boldsymbol{q}=\left(x_{1}, y, z\right)$ in such an open ball, and let $\boldsymbol{L}$ be the straight line segment from $\boldsymbol{s}$ to $\boldsymbol{q}$. Then, of course, $\boldsymbol{L}$ lies in $\boldsymbol{D}$. Now let's integrate $\boldsymbol{F}$ from $\boldsymbol{p}$ to $\boldsymbol{s}$ by going along any curve $\boldsymbol{C}$ from $\boldsymbol{p}$ to $\boldsymbol{q}$ and then along $\boldsymbol{L}$ from $\boldsymbol{q}$ to $s$ :

$$
g(\boldsymbol{s})=g(x, y, z)=\int_{C} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}+\int_{L} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r} .
$$

The first integral on the right does not depend on $x$, and so $\frac{\partial}{\partial x} \int_{\boldsymbol{C}} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}=0$. Thus

$$
\frac{\partial g}{\partial x}=\frac{\partial}{\partial x} \int_{L} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}
$$

We clearly need to find $\int_{L} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}$. This is easy. Suppose

$$
\boldsymbol{F}(\boldsymbol{r})=f_{1}(\boldsymbol{r}) \boldsymbol{i}+f_{2}(\boldsymbol{r}) \boldsymbol{j}+f_{3}(\boldsymbol{r}) \boldsymbol{k}
$$

A vector description of $\boldsymbol{L}$ is simply $\boldsymbol{r}(t)=\boldsymbol{i}+y \boldsymbol{j}+z \boldsymbol{k}, x_{1} \leq t \leq x$. Thus $\frac{d \boldsymbol{r}}{d t}=\boldsymbol{i}$, and our line integral becomes simply $\int_{L} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}=\int_{x_{1}}^{x} f_{1}(t, y, z) d t$. We are almost done, for note that now

$$
\frac{\partial}{\partial x} \int_{L} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}=\frac{\partial}{\partial x} \int_{x_{1}}^{x} f_{1}(t, y, z) d t=f_{1}(x, y, z)
$$

Hence

$$
\frac{\partial g}{\partial x}=f_{1} .
$$

It should be clear to one and all how to show that $\frac{\partial g}{\partial y}=f_{2}$ and $\frac{\partial g}{\partial z}=f_{3}$, thus giving us the desired result: $\boldsymbol{F}=\nabla g$.

## Exercises

12. Prove that $\frac{\partial g}{\partial y}=f_{2}$, where $g$ and $f_{2}$ are as in the preceding discussion.
13. Prove that if $\boldsymbol{F}: \boldsymbol{D} \rightarrow \boldsymbol{R}^{3}$, where $\boldsymbol{D}$ is open and connected, and every $\int_{\boldsymbol{C}} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}$ is path independent, then $\oint_{P} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}=0$ for every closed path in $\boldsymbol{D}$. ( A closed path, or
curve, is one with no endpoints.) [Physicists and others like to use a snake sign with a little circle superimposed on it $\oint$ to indicate that the path of integration is closed.]
14. Prove that if $\boldsymbol{F}: \boldsymbol{D} \rightarrow \boldsymbol{R}^{3}$, where $\boldsymbol{D}$ is open and connected, and $\oint_{\boldsymbol{P}} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}=0$ for every closed path in $\boldsymbol{D}$, then every $\int_{\boldsymbol{C}} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}$ is path independent.
15. a)Find a potential function $g$ for the function $\boldsymbol{F}(\boldsymbol{r})=y z \boldsymbol{i}+x z \boldsymbol{j}+x y \boldsymbol{k}$.
b)Evaluate the line integral $\int_{\boldsymbol{C}} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}$, where $\boldsymbol{C}$ is the curve

$$
\boldsymbol{r}(t)=\left(e^{t} \sin t\right) \boldsymbol{i}+t^{2} e^{3 t} \boldsymbol{j}+\cos ^{3} t \boldsymbol{k}, 0 \leq t \leq 1 .
$$

16. a)Find a potential function $g$ for the function $\boldsymbol{F}(\boldsymbol{r})=e^{y+2 z}(\boldsymbol{i}+x \boldsymbol{j}+2 x \boldsymbol{k})$.
b) Find another potential function for $\boldsymbol{F}$ in part a).
b)Evaluate the line integral $\int_{C} \boldsymbol{F}(\boldsymbol{r}) \cdot d \boldsymbol{r}$, where $\boldsymbol{C}$ is the curve

$$
\boldsymbol{r}(t)=t \cos 2 t^{2} \boldsymbol{i}+4 t \boldsymbol{j}+e^{2 t} \boldsymbol{k}, 0 \leq t \leq \sqrt{\pi} .
$$

17. Evaluate $\oint_{\boldsymbol{E}}\left[\left(e^{x} \sin y+3 y\right) \boldsymbol{i}+\left(e^{x} \cos y+2 x-2 y\right) \boldsymbol{j}\right] \cdot d \boldsymbol{r} \quad$ where $\boldsymbol{E}$ is the ellipse $4 x^{2}+y^{2}=4$ oriented clockwise.
[Really good hint: Find the gradient of $g(x, y, z)=e^{x} \sin y+2 x y-y^{2}$.]
