Using the Continuity Applet to Find δ Values

These notes concern using the continuity applet to study the ϵ-δ definition of continuity, and in particular, to measure “how continuous” continuous functions are.

Recall the following definition: Given a function f and a point x_0, we define for each $\epsilon > 0$

$$\delta(\epsilon; x_0, f) = \text{l.u.b}\{\delta \quad \text{such that} \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon\}$$

We computed this explicitly – in the noteson continuity – for $f(x) = 1/x$, but it was a lot of work. This project carries this investigation forward on the computer, where it is not a lot of work.

Actually, we will work with a slightly version of this: We will only consider values of ϵ and δ that are inverse powers of 2. That is,

$$\delta = 2^{-m} \qquad \text{and} \qquad \epsilon = 2^{-n}$$

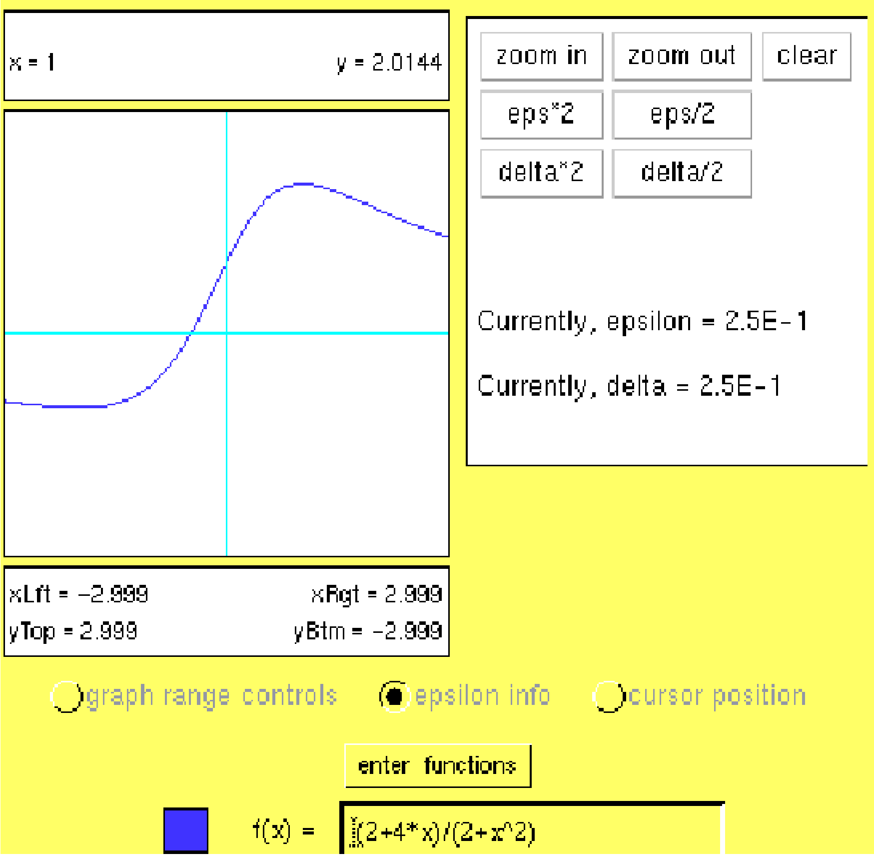
Definition: For f and x_0 fixed we define $m(n; x_0, f)$ to be the **smallest** value if m so that

$$|x - x_0| < 2^{-m} \Rightarrow |f(x) - f(x_0)| < 2^{-n} \ .$$

In other words, if ϵ, the desired degree of output accuracy, is set to be 2^{-n}, then for the function f at the point x_0, $\delta = 2^{-m(n;x_0,f)}$.

This has the following meaning: If $|f(x) - f(x_0)| < 2^{-n}$, then when written out in binary, the first n bits of $f(x)$ and $f(x_0)$ will be the same. So $m(n; x_0, f)$ is the number of leading bits of x and x_0 that have to be the same to ensure this level of accuracy. In other words, if we want n accurate bits in the output at x_0, we need $m(n; x_0, f)$ accurate bits of input. So figuring out what $m(n; x_0, f)$ is is a very practical matter.

That is what we are going to do using a java applet. The link to the applet is on the page that linked to this project description. If you go to it, you see the following:



x = 1
y = 2.0144

Currently, epsilon = 2.5E-1

Currently, delta = 2.5E-1

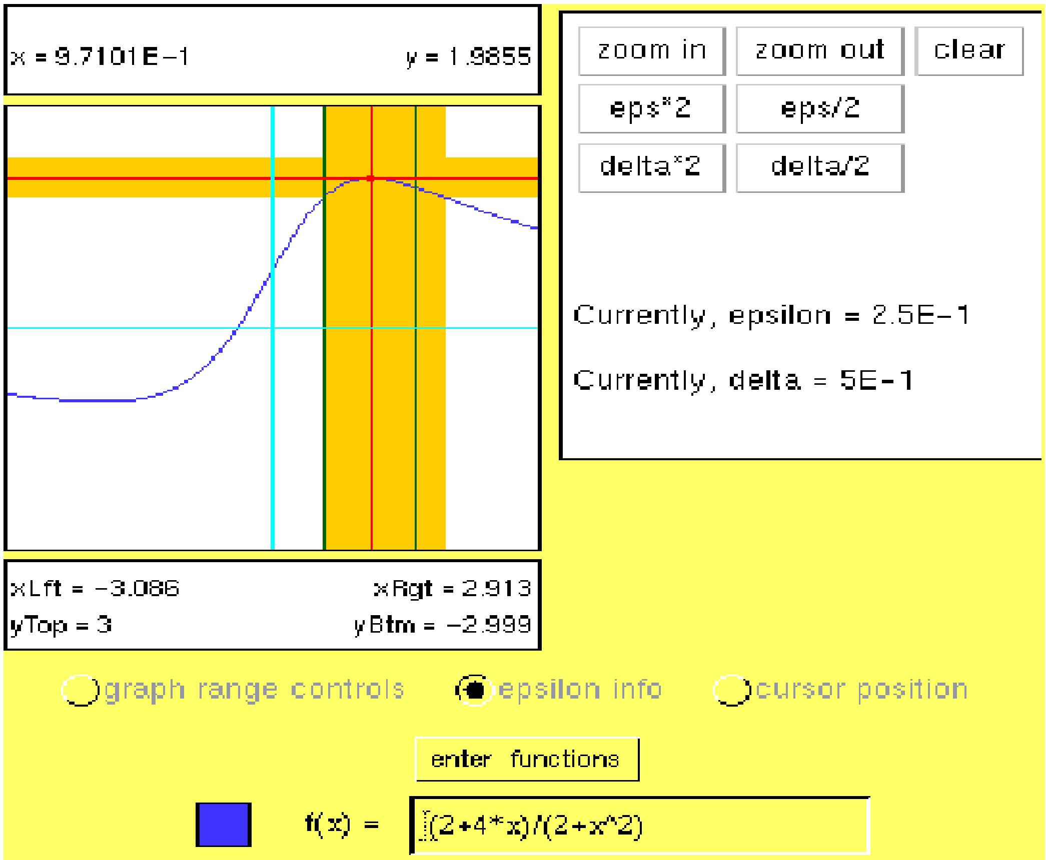
xLft = -2.999
xRgt = 2.999
yTop = 2.999
yBtm = -2.999

○ graph range controls ● epsilon info ○ cursor position

f(x) = (2+4*x)/(2+x^2)

At least after you have clicked on the radio button labeled "epsilon info". Move your cursor over the graph. Notice that the x and y coordinates of the cursor show up in the box above the graph.

Move the cursor until x is a as close to 1.0 as you can get it and click. You will see:



x = 9.7101E-1
y = 1.9855

Currently, epsilon = 2.5E-1

Currently, delta = 5E-1

xLft = -3.086
xRgt = 2.913
yTop = 3
yBtm = -2.999

○ graph range controls ● epsilon info ○ cursor position

f(x) = (2+4*x)/(2+x^2)

The default function the applet starts up with is

$$f(x) = \frac{2+4x}{2+x^2}\ .$$

This can be changed by entering a new function in the panel at the bottom, but for now, let's stick with this one.

The red vertical line runs through the x value, x_0 at the point you clicked on – in this case 1. The red horizontal line runs though the y value $y = f(x_0)$ – in this case $f(1) = 2$.

The horizontal orange band covers all of the y values that are within ϵ of $y = f(x_0)$, our desired output. This is the region of outputs that are “within tolerances”. The vertical orange band runs through all x values that produce outputs y in the tolerance band; i.e, with $|y - f(x_0)| < \epsilon$. Initially, $\epsilon = 0.25 = 2^{-2}$, as you see in the panel on the right. You can double or halve this value by pressing the corresponding buttons on the panel, so this is under your control in the experiments to follow.

The two green vertical lines run through are given by

$$x = x_0 - \delta \qquad \text{and} \qquad x = x_0 + \delta \ .$$

Again, the default value is $\delta = 0.25 = 2^{-2}$, but you can halve it or double it by pressing the buttons on the right.

Here is the point:

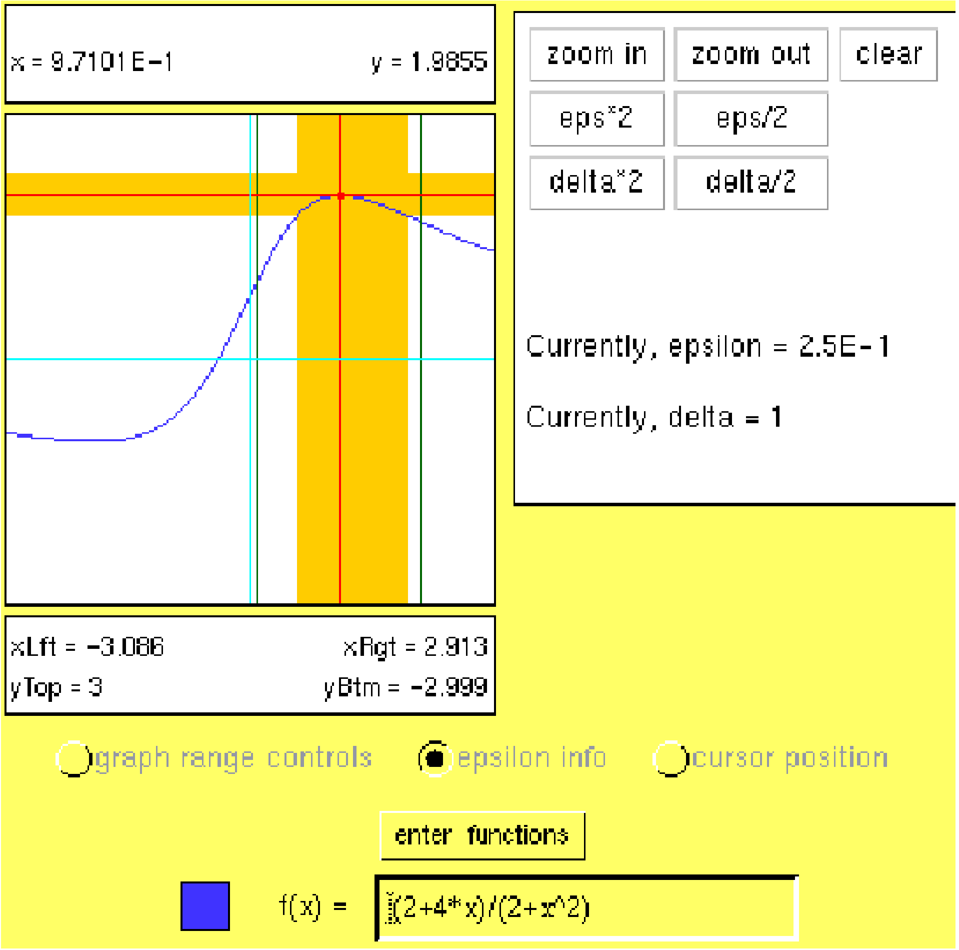
If these two green lines are inside the vertical orange band. then for this δ and ϵ, we have

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

since every x such that $|x - x_0| < \delta$ is in the vertical orange region, which means that the corresponding y values lie in the tolerance band; i.e, that $|f(x) - f(x_0)| < \epsilon$.

Let us use this to find $m(n; 1, f)$ for several values of n. The first thing we will do is to adjust δ, and see how big a value we can get away with. Push the ”delta*2” button a couple of times. Each time you do, the green lines move out, and the region of x satisfying $|x - x_0| < \delta$ grows with δ.

For $\delta = 1.0$, here is what you see.



x = 9.7101E-1 y = 1.9855

Currently, epsilon = 2.5E-1

Currently, delta = 1

xLft = -3.086 xRgt = 2.913

yTop = 3 yBtm = -2.999

○ graph range controls ● epsilon info ○ cursor position

f(x) = (2+4*x)/(2+x^2)

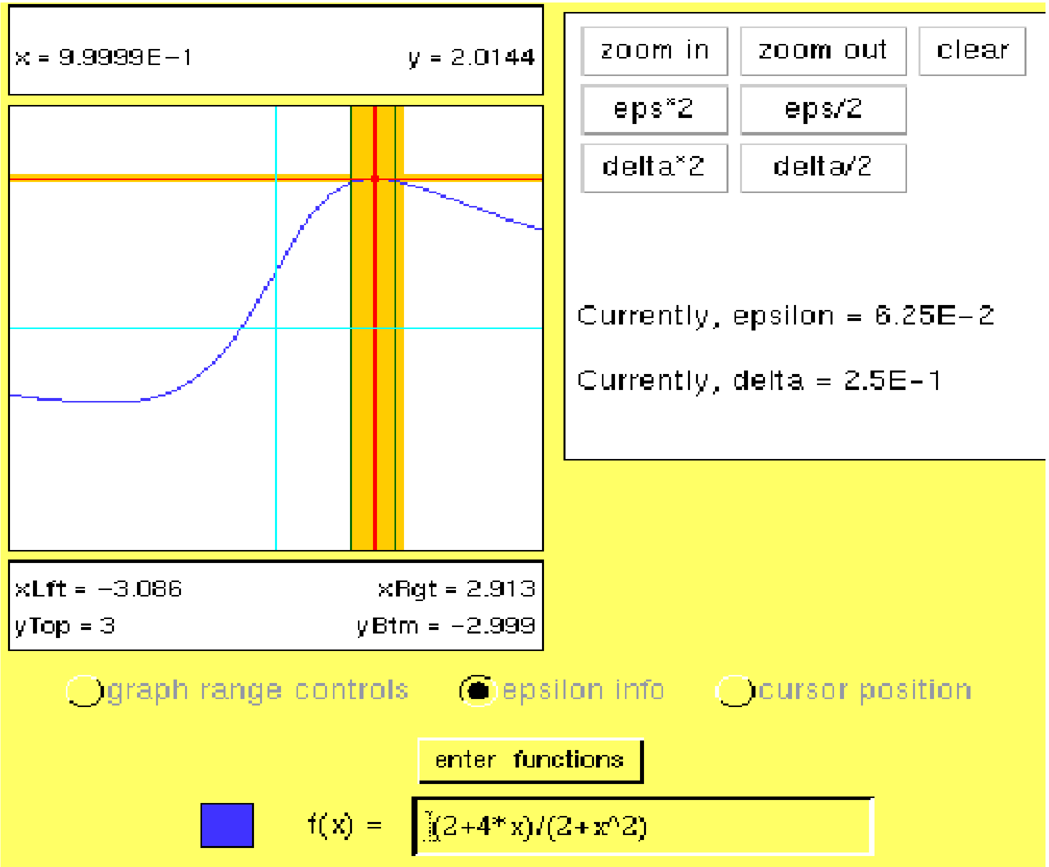
Notice that this value is too big – the green lines are outside the vertical orange band. But for $\delta = 0.5$, everything is O.K., as we saw in the previous picture.

The conclusion is that for $\epsilon = 0.25 = 2^{-2}$, i.e., with $n = 2$, we can take $m = 1$ and stay in tollerance, but not $m = 0$. So

$$m(2; 1, f) = 1 \ .$$

Now lets try a smaller value of ϵ, $\epsilon = 0.0625 = 2^{-4}$.

Here is the picture for that:



x = 9.9999E-1 y = 2.0144

Currently, epsilon = 6.25E-2

Currently, delta = 2.5E-1

xLft = -3.086 xRgt = 2.913

yTop = 3 yBtm = -2.999

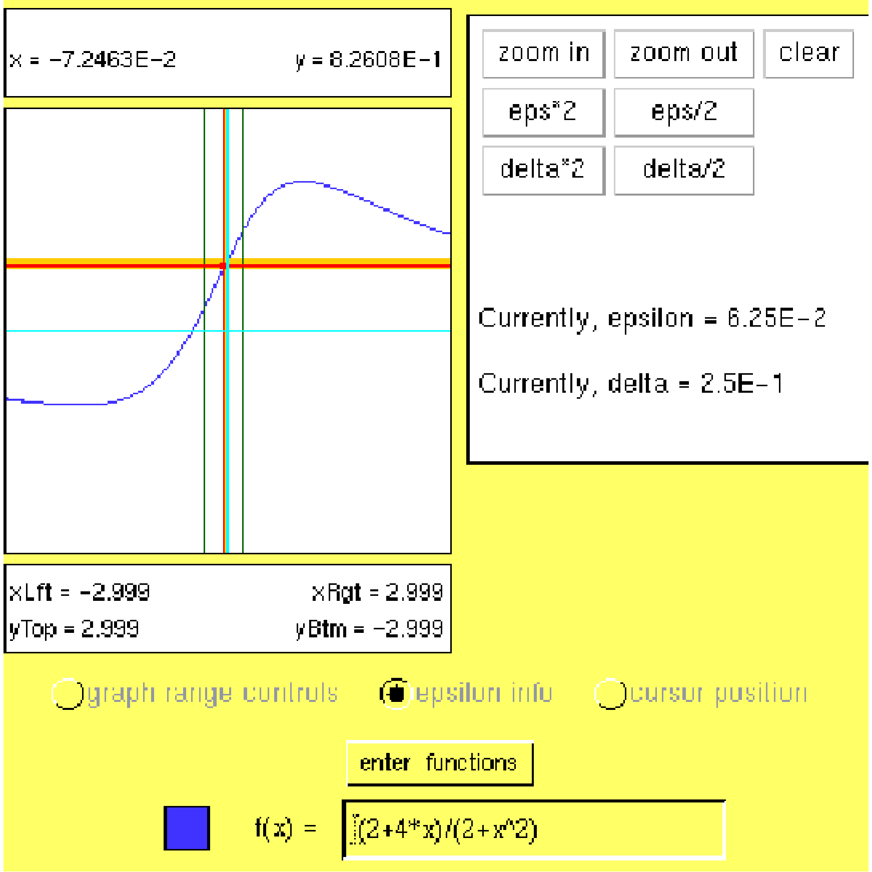
f(x) = (2+4*x)/(2+x^2)

The value of δ has been adjusted to bring the green lines just inside the vertical orange band. Notice that this happens for $\delta = 0.25 = 2^{-2}$ so

$$m(4; 1, f) = 2$$

So in this case, only one more bit of accuracy on the input bought us two more bits of accuracy on the output! That's great – but is it always so good? Let's see; let's try another value of x_0 other than 1. Let's look at $x_0 = 0$. We'll use the same value of ϵ namely $0.0625 = 2^{-4}$.

Here is the picture:



x = -7.2463E-2 y = 8.2608E-1

xLft = -2.999 xRgt = 2.999
yTop = 2.999 yBtm = -2.999

Currently, epsilon = 6.25E-2

Currently, delta = 2.5E-1

○ graph range controls ● epsilon info ○ cursor position

f(x) = (2+4*x)/(2+x^2)

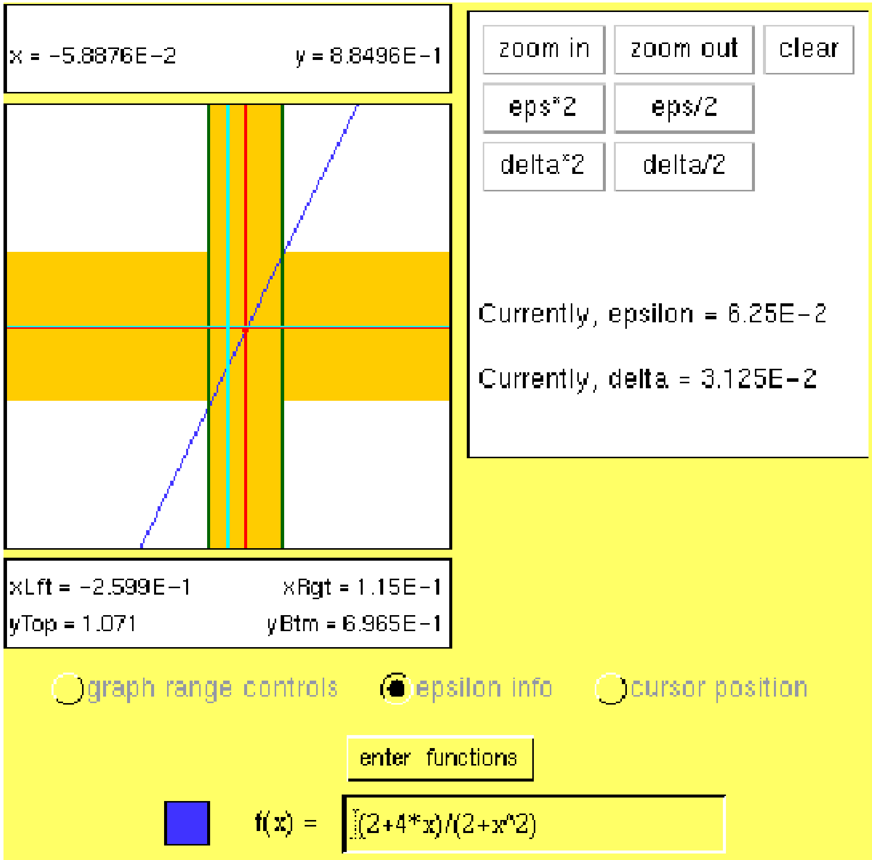
Clearly at $x_0 = 0$ and with $0.0625 = 2^{-4}$, we need a much smaller δ. As it is, the green lines are way out of bounds.

In fact, to see what's going on we'll have to "zoom in".

To do this, click down on the mouse away from the red dot, drag over to the red dot, and release the mouse button. The graph will be redrawn with the red dot at the center. Actually, to keep it from getting covered up by the cross-hairs, it is better to release just a bit away. Wherever you release, provided you dragged the mouse for more than three pixels, becomes the new center.

Now zoom in by pressing the "zoom in" button a couple of times.

Here is what you see after doing this, and then decreasing δ until the green lines just fit inside the orange band:



x = -5.8876E-2 y = 8.8496E-1

Currently, epsilon = 6.25E-2

Currently, delta = 3.125E-2

xLft = -2.599E-1 xRgt = 1.15E-1
yTop = 1.071 yBtm = 6.965E-1

○ graph range controls ● epsilon info ○ cursor position

f(x) = (2+4*x)/(2+x^2)

Clearly in this case we need $\delta = 0.0325 = 2^{-5}$ in order to work with $\epsilon = 0.0625 = 2^{-4}$. That is, $m(4; 0, f) = 5$.

Notice also that when we zoomed in, the part of the graph of f that we're working with here looks pretty linear. In fact we could have computed the relation between δ and ϵ at $x_0 = 1$ by using the tangent line to the graph of f at this point instead of f itself, and we'd have gotten the same answer – since the graphs of f and its tangent line are evidently essentially the same thing here. This should suggest to you that finding tangent linr slopes provides a shortcut for computing δ values!