Continuity of Functions and Limits

The functions we are concerned with in applications of mathematics describe how real output depends on real input – say say how much current one gets out of an amplifier for a given applied voltage. If $f(x)$ denotes the output current when the input voltage is x, then one thing we might want to know is what voltage x_0 we should apply to produce a current of, say, 5 amps. That is, we would like to find the value of x_0 so that

$$f(x_0) = 5 \ . \tag{1}$$

However, this is not really enough in any application. This is simply because in the real world, we will never be able to adjust the applied voltage to be exactly x_0 volts – there will always be some discrepency. The best we can hope to do is to keep the applied voltage x in some small interval about x_0:

$$x_0 - \delta < x_0 < x_0 + \delta \tag{2}$$

for some small but positive number δ. The smaller we want to keep δ, the harder we will have to work to keep the applied voltage in such a range. How small is enough?

That depends on the application – that is on how close to 5 amps we need the output current, $f(x)$, to be. Suppose our application stipulates a tolerance of no more than $\pm\epsilon$ amps for some small positive number ϵ. That is, our application demands that we keep

$$5 - \epsilon < f(x) < 5 + \epsilon$$

or, what is the same thing given that $f(x_0) = 5$,

$$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon \ . \tag{3}$$

The question now is: can we choose δ small enough – but still positive – so that (3) is satisfied by all x that satisfy (2).

Clearly, the δ we need in (2) depends on the epsilon stipulated in (3); generally, the smaller ϵ is, the smaller δ must be. When for any given ϵ, we can choose a corresponding $\delta > 0$ so that all x satisfying (2) satisfy (3), the function f represents represents a "controllable" process or operation. You may have to work very hard to keep your inputs in a narrow interval, but if you do, you know your outputs will also be in a narrow interval – as narrow as you need.

Not all functions have this property. Before giving examples, we name it in a formal definition.

Definition of Continuity We say a function is continuous at a point x_0 in its domain if for any $\epsilon > 0$ there is a corresponding $\delta > 0$ so that (3) holds for all x such that (2) holds. We say that a function f is continuous if it is continuous at all points in its domain. If this property fails to hold at some point, we say the function is discontinuous, and, more specifically, is discontinuous at that point, or has a discontinuity at that point.

Another more compact way of writing the the continuity condition is that for each $\epsilon > 0$, there is a $\delta > 0$ so that

$$|x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \epsilon \ .$$

Now, lets discuss some example of discontinuous functions. First consider the **sign function**, $\mathrm{sgn}(x)$ which is defined by

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x = 0 \\ 1 & \text{for} \quad x > 0 \end{cases}$$

That is, it gives the sign of x.

The sign function $\mathrm{sgn}(x)$ is discontinuous at 0: Every small interval about zero contains positive numbers x, for which $\mathrm{sgn}(x) = 1$, and negative numbers x for which which $\mathrm{sgn}(x) = -1$. Therefore, if $\epsilon = 1/2$, there is no $\delta > 0$ so that

$$-\delta < x < \delta$$

implies

$$-1/2 < \mathrm{sgn}(x) < 1/2$$

which is what (2) and (3) become with $x_0 = 0$, $f(x) = \mathrm{sgn}(x)$ and $\epsilon = 1/2$ since, by definition, $\mathrm{sgn}(0) = 0$.

Next, consider the Heaviside function $\mathrm{H}(x)$ which is defined by

$$\mathrm{H}(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x \geq 0 \end{cases}$$

is discontinuous at $x = 0$. This time you provide the proof – that is display an ϵ for which no corresponding δ can be found.

How about some examples of continuous functions? First, consider the constant functions: Let a be any fixed number, and suppose $f(x) = a$ for all x. This is continuous since no matter how small ϵ is, (3) holds whenever (2) does for any positive δ at all – the output of this function is always exactly a, so there is no need to control the input.

For a less trivial example, let's consider $f(x) = x$. In this case, we can take $\delta = \epsilon$, and then (2) implies (3). So this function is continuous.

Now here is something a bit more complicated: $f(x) = 1/x$ on the domain $x \neq 0$. This function is also continuous, but a bit more work is required. Pick some positive number x_0, and suppose that (2) holds. We will suppose further that $\delta < x_0$ so that the interval in (2) is contained in the domain of f. Then for any x in this interval

$$f(x_0 + \delta) < f(x_0) < f(x_0 - \delta)$$

because f is a monotone decreasing function on the positive real numbers – that is; whenever $x > y > 0$, $0 < f(y) < f(x)$.

Now clearly (3) will be satisfied if both

$$f(x_0) - \epsilon \leq f(x_0 + \delta) \qquad \text{and} \qquad f(x_0 - \delta) \leq f(x_0) + \epsilon \tag{5}$$

which is the same as

$$1/x_0 - \epsilon \leq 1/(x_0 + \delta) \qquad \text{and} \qquad 1/(x_0 - \delta) \leq 1/x_0 + \epsilon$$

Let's focus on these one at a time. Clearly, when δ is very small,

$$\frac{1}{x_0 - \delta} \leq \frac{1}{x_0} + \epsilon \; . \tag{6}$$

Let's find the value δ_1 of δ that makes the two sides equal. Any δ smaller than that will make (6) true. Setting both sides equal and solving for δ, one easily finds

$$\delta_1 = \frac{\epsilon x_0^2}{1 + \epsilon x_0} \; . \tag{7}$$

Similarly, we look for the value δ_2 of δ that makes produces equality in

$$\frac{1}{x_0} - \epsilon \leq \frac{1}{x_0 + \delta} \tag{8}$$

which clearly holds for δ sufficiently close to zero. In fact, it is true for any positive value of δ if $\epsilon > 1/x_0$, so suppose this isn't the case. Then, Solving for δ_2 we find

$$\delta_2 = \frac{\epsilon x_0^2}{1 - \epsilon x_0} \; .$$

Clearly, $\delta_1 < \delta_2$. So as long as $\delta \leq \delta_1$ with δ_1 given in (7), (6) and (8) both hold, and so (5) holds. So we found our δ, and have proved that $f(x) = 1/x$ is continuous on $x > 0$.

We've actually done a bit more: We found the larges possible δ that makes (2) imply (3) for this function at the point x_0¿

Given any function f that is continuous at x_0, define

$$\delta(\epsilon; x_0, f) = \text{l.u.b}\{\delta \text{ such that (2) implies (3)}\}$$

The work we just did above shows that for $f(x) = 1/x$,

$$\delta(\epsilon; x_0, f) = \frac{\epsilon x_0^2}{1 + \epsilon x_0} \; .$$

To see what this means, suppose we we want an output of 100 from $f(x) = 1/x$. Then we should put in $x_0 = 0.01$. If we insist on 1% accuracy, that is that our out put lie between 99 and 101, how close to $x_0 = 0.01$ do we need our input to be?

Well, in this case, since we are asking for $99 < f(x) < 101$, we are taking $\epsilon = 1$. Also x_0 is 0.01. Plugging these in gives us

$$\delta(1; 0.01, f) = \frac{0.0001}{1.01} \; .$$

That is we need to control our input to four decimal places of accuracy just ot get two decimal places of accuracy on the output.

the validity of (2) implies that of (3). Thus, f is continuous at all positive numbers x_0. A similar proof, left to the reader, works for all negative numbers. Thus, f is continuous.

Notice in this example that the value of δ depends on x_0 as well as on ϵ – at some points this operation is more easily "controlled" than at others.

Now consider

$$f(x) = \frac{1+x^2}{1+x^4} \ .$$

Instead of tackling this directly, by computing the required δ in terms of ϵ and x_0 as we did above, we will show this function to be continuous by disassembling it into its constituent operations, and analysing those separately. In terms of the three functions

$$g(x) = 1 \qquad h(x) = x \qquad \text{and} \qquad k(x) = 1/x$$

we have

$$f = (g + h * h) * k \circ (g + h * h * h * h) \ .$$

We have already seen that each of the "building blocks" g, h and k are continuous. The next theorem allows us to conclude from this that f is continuous as well – without having to explicitly find the δ.

Theorem *Let f and g be any two continuous functions on the same domain. Then*

$$f + g \qquad f - g \qquad \text{and} \quad f * g$$

are continuous. Furthermore, at all points x_0 in the domain at which $g(x_0) \neq 0$, f/g is continuous. Finally, if the range of f is contained in the domain of g, then $g \circ f$ is continuous on the domain of f.

The proof of this is very simple. Consider the case of $f+g$ first. Pick any *epsilon* > 0, and any x_0 in the domain. By hypothesis, f and g are continuous, so there is a δ_1 so that

$$f(x_0) - \epsilon/2 < f(x) < f(x_0) + \epsilon/2$$

whenever

$$x_0 - \delta_1 < x < x_0 + \delta_1 \ .$$

Likewise, there is a $\delta_2 > 0$ so that

$$g(x_0) - \epsilon/2 < g(x) < g(x_0) + \epsilon/2$$

whenever

$$x_0 - \delta_2 < x < x_0 + \delta_2 \ .$$

Now simply take $\delta = \min\{\delta_1, \delta_2\}$, and we have

$$f(x_0) + g(x_0) - \epsilon < f(x) + g(x) < f(x_0) + g(x_0) + \epsilon$$

whenever

$$x_0 - \delta < x < x_0 + \delta \ .$$

The cases of subtraction, multiplication and division are similar, and are left to the reader as exercises.

The case of composition is more interesting. Pick any $\epsilon > 0$, and any x_0. Let $y_0 = f(x_0)$. By hypothesis, g is continuous at y_0, so there is a $\delta_1 > 0$ with

$$g(y_0) - \epsilon < g(y) < g(y_0) + \epsilon$$

whenever

$$y_0 - \delta_1 < y < y_0 + \delta_1 \ .$$

Replacing y_0 with $f(x_0)$ and y with $f(x)$, this becomes

$$g \circ f(x_0) - \epsilon < g \circ f(x) < g \circ f(x_0) + \epsilon$$

whenever

$$f(x_0) - \delta_1 < f(x) < f(x_0) + \delta_1 \ .$$

Now just think of δ_1 as an ϵ, and by the continuity of f, we can find a $\delta > 0$ so that

$$f(x_0) - \delta_1 < f(x) < f(x_0) + \delta_1$$

whenever

$$x_0 - \delta < x < x_0 + \delta \ .$$

Putting it altogether, we have that

$$g \circ f(x_0) - \epsilon < g \circ f(x) < g \circ f(x_0) + \epsilon$$

whenever

$$x_0 - \delta < x < x_0 + \delta \ ,$$

and this proves the continuity of $g \circ f$.

So the main strategy for investigating continuity of a function is to take it apart into an "assembly line" of simple pieces, and to see what inputs run into discontinuities of the individual pieces somewhere down the assembly line. Any input x_0 which doesn't lead to a discontinuity of any stage on the assembly line is a point of continuity of the function.

To use this, we need to know something about the simple "building block" functions. Lets collect our examples into a theorem:

Theorem *The following functions are continuous on their stated domains:*

(a) $f(x) = a$, a some constant, for all real x

(b) $g(x) = x$, for all real x

(c) $h(x) = 1/x$, for all $x \neq 0$

Now if $p(x)$ is any polynomial, $p(x)$ is built up out of f and g by multiplication and addition, so an immediate consequence is that all polynomials are continuous. Moreover, if $p(x)$ and $q(x)$ are any two polynomials, let $r(x) = p(x)/q(x)$ where the domain is all points x such that $q(x) \neq 0$ – so the division makes sense. Such functions, being the ratio of two polynomials, are called rational functions. Applying our theorem again, we see that all rational functions are continuous everywhere in their domains.

Rational functions are all we can build out of the building blocks provided in Theorem 2. To go further, we need to add more building blocks to our repitoire.

Theorem *The following functions are continuous on the whole real line:*

$$\sin(x) \qquad \cos(x)\ .$$

Proof: Since $\cos(x) = \sin(x + \pi/2)$, it suffices to show that $\sin(x)$ is continuous: we already know that $f(x) = x + \pi/2$ is continuous, and then composition with sin gives us $\cos(x)$.

To prove that $\sin(x)$ is continuous, we have to go back to the definition. So pick an $\epsilon > 0$ and an x_0. Let $h = x - x_0$. We are interested in the situation that x is close to x_0, so we may as well assume that $|h| < \pi/2$. Then form the angle addition formula for sin,

$$\sin(x) = \sin(x_0 + h) = \sin(x_0)\cos(h) + \cos(x_0)\sin(h)\ ,$$

so that

$$\sin(x) - \sin(x_0) = \sin(x_0)(\cos(h) - 1) + \cos(x_0)\sin(h)\ .$$

Since $|\cos(\theta)| \leq 1$ and $|\sin(\theta)| \leq 1$ for any θ,

$$|\sin(x) - \sin(x_0)| \leq |\cos(h) - 1| + |\sin(h)|\ .$$

But for any $h > 0$, $\sin(h) \leq h$, so that

$$|\sin(h)| \leq |h|\ .$$

Then

$$1 - \cos(h) = \frac{(1 - \cos(h))(1 + \cos(h))}{1 + \cos(h)} = \frac{\sin^2(h)}{1 + \cos(h)} \leq h^2$$

for all h with $|h| \leq \pi/2$, since for such h, $\cos(h) \geq 0$. Hence

$$|\sin(x) - \sin(x_0)| \leq h^2 + h \leq (\pi/2 + 1)h$$

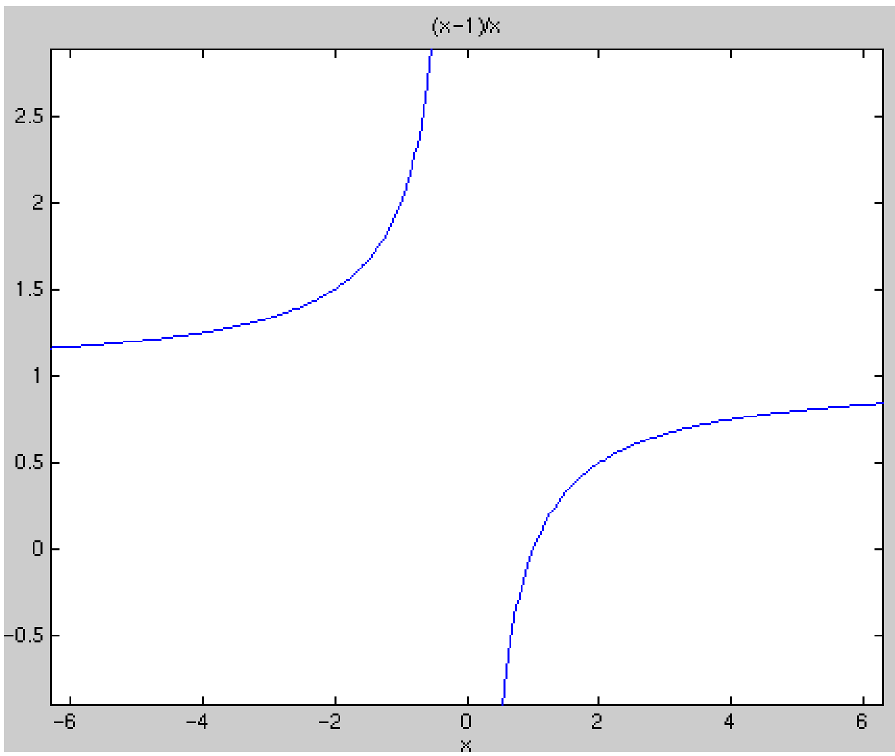
for all $h \leq \pi/2$. So given $\epsilon > 0$, let $\delta = \epsilon/(1+\pi/2)$. By the above, recalling that $h = x - x_0$, whenever $|x - x_0| < \delta$, $|\sin(x) - \sin(x_0)| < \epsilon$. This proves the continuity of $\sin(x)$.

We can now build up a large library of known continuous functions by combining rational functions of x with trigonometric functions. Being able to recognize continuous functions is very useful when we compute limits – the next subject.

To introduce the notion of limits, consider two examples. First, let $f(x) = 1/(1-x)$, $x \neq 1$. Then f is continuous. Composing it with itself to define a new function $g(x)$, we get

$$g(x) = f \circ f(x) = \frac{x-1}{x} \qquad x \neq 1, x \neq 0$$

The point 0 had to be excluded from the domain since $f(0) = 1$ which isn't in the domain of the second f in our composition. However, the two excluded points are very different. Here is a graph of the function:



You see there is a big problem at $x = 0$, but no problem at $x = 1$. We could "paper over" this gap in the domain by defining $g(1) = 0$. **The resulting function on the larger domain is still continuous!**

On the other hand, there is no way to define $g(0)$ to make the function continuous at 0.

We will give a precise definition of limits, and introduce some notation for discussing them in a short while, but looking ahead a bit, the way we'd describe this situation is

$$\lim_{x \to 1} g(x) = 0$$

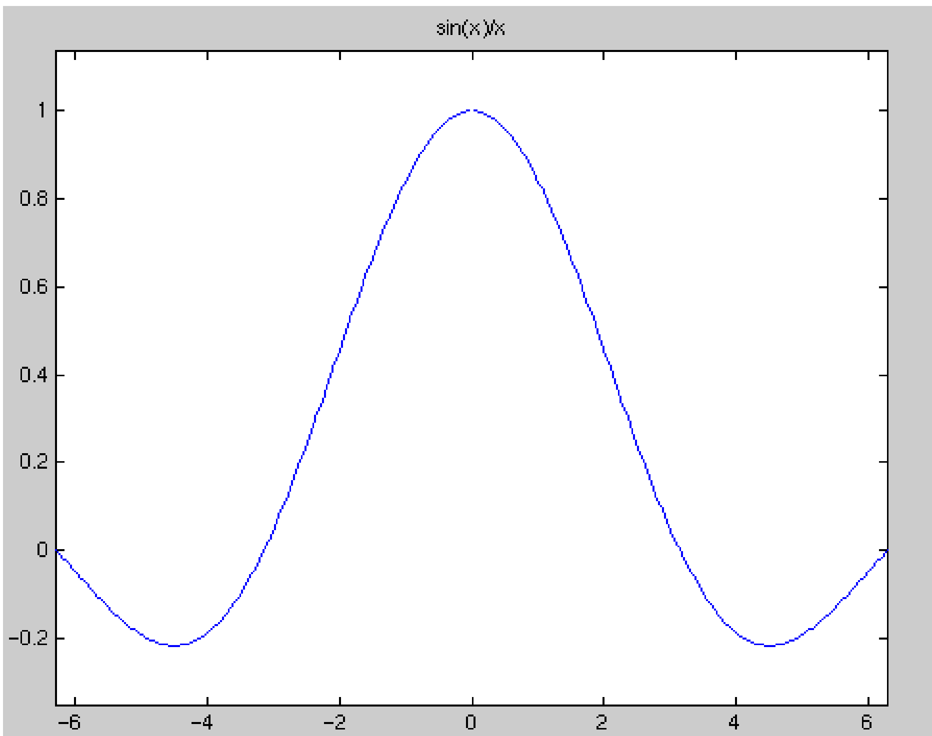
and

$$\lim_{x \to 0} g(x) \qquad \text{does not exist .}$$

Before giving the definitions, let's look at another example. This time, take

$$f(x) = \frac{\sin x}{x} \qquad x \neq 0 \; .$$

This is continuous since it is the ratio of two continuous functions restricted to where the denominator is never zero. It is not defined at $x = 0$, but if we graph it, here is what we see:



There is no problem visible at $x = 0$. In fact, as we will see, if we define $f(0) = 1$, so that now f is defined everywhere, the resulting function is continuous. In the notation to be introduced shortly, this is

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \; .$$

Definition Let f be a function and x_0 a real number so that for all $\delta > 0$, the interval $(x_0 - \delta, x_0 + \delta)$ contains points in the domain of f. We say that L is the **limit of** f **as** x **approached** x_0 in case the function $g(x)$ given by

$$g(x) = \begin{cases} f(x) & \text{for} \quad x \neq x_0 \\ L & \text{for} \quad x = x_0 \end{cases}$$

is continuous. In this case we write

$$\lim_{x \to x_0} f(x) = L \ .$$

The definition says nothing about whether or not x_0 is already in the domain or not.

Notice that by the definition of continuity, g being continuous means that for every $\epsilon > 0$, there is a $\delta > 0$ so that $|x - x_0| < \delta$ implies $|g(x) - g(x_0)| < \epsilon$. but since $g(x_0) = L$, and since ofr all other x, $g(x) = f(x)$, this is the same as having

$$|x - x_0| < \delta \to |f(x) - L| < \epsilon$$

for all $x \neq x_0$ in the domain of f. This equivalent formulation is often taken as the definition of limiting values. What it says in plain words is that *as the input x is "squeezed down" around x_0, the resulting output "squeezes down" around L.*

Theorem If a function f is continuous at a point x_0 in its domain, then

$$\lim_{x \to x_0} f(x) = f(x_0) \ .$$

This theorem says that limits of continuous can be computed by evaluation. This is the good case, and it is something you can't do to compute $\lim_{x \to 0} \sin x/x$ since you can't even evaluate this at 0. But for example we know that $f(x) = 3x^4 + 2x - 1$ is continuous everywhere – all polynomials are – so

$$\lim_{x \to 1} f(x) = f(1) = 4 \ .$$

The proof of the theorem is very simple. To compute a limit means to find a value L that we can assign to f at x_0 to make it continuous there. But since it is already continuous, we just use the value it already has.

The next theorem allows us to "divide and conquer" in the computation of limits.

Theorem Suppose that $\lim_{x \to x_0} f(x) = L$ and that $g(x)$ is continuous at L. Then

$$\lim_{x \to x_0} g(f(x)) = g(L) \ .$$

bigskip

For example, consider $\lim_{x \to 0}(1 - \cos^2 x)/x^2$. Since

$$\frac{1 - \cos^2 x}{x^2} = \left(\frac{\sin x}{x}\right)^2$$

$$\lim_{x\to 0}\frac{1-\cos^2 x}{x^2} = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = (1)^2 = 1$$

will follow as soon as we know $lim_{x\to 0} sinx/x = 1$, as asserted above.

The proof of this theorem is just as simple: Assigning the value L to f at x_0 make f continuous there. By hypothesis, g is already continuous, and we know that the composition of continuous functions is continuous. So $g \circ f$ is continuous at x_0 when $f(x_0)$ is defined to be L. But then the value of $g \circ f$ at x_0 is $g(L)$. This being the unique value that makes $g \circ f$ continuous at x_0, this is the limit.

Similar arguments yields the following theorem:

Theorem Suppose f and g two functions such that

$$\lim_{x\to x_0} f(x) = L \qquad \text{amd} \qquad \lim_{x\to x_0} g(x) = M$$

Then

$$\lim_{x\to x_0}\big(f+g\big)(x) = L + M$$
$$\lim_{x\to x_0}\big(fg\big)(x) = LM$$
$$\lim_{x\to x_0}\left(\frac{f}{g}\right)(x) = \frac{L}{M}$$

where in the last case only we also require $M \neq 0$.

In words this says that the limit of a sum is the sum of the limits, that the limit of a product is the product of the limits, and that the limit of a quotient is the quotient of the limits, provided that the denominator is not zero.

As an example, consider

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

Notice that

$$\frac{1-\cos x}{x^2} = \frac{1-\cos x}{x^2}\frac{1+\cos x}{1+\cos x} = \frac{\sin^2 x}{x^2}\frac{1}{1+\cos x}$$

Since we know that

$$\lim_{x\to 0}\frac{\sin^2 x}{x^2} = 1$$

and since $1/(1+\cos x)$ is continuous at 0, so that

$$\lim_{x\to 0}\frac{1}{1+\cos x} = \frac{1}{2}$$

it follows that

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = 1\frac{1}{2} = \frac{1}{2}\ .$$

This divide and conquer strategy is how we will compute most limits. But a few basic cases like $\sin x/x$ can't be further broken down, and these we have to do "by hand".

A proof of

$$\lim_{x\to 0}\frac{\sin x}{x}$$

can be found in the text. I'll add one here after I get the pictures needed for it produced.