

SOME MATRIX REARRANGEMENT INEQUALITIES

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Dedicated to Professor Roberto Conti

Abstract

We investigate a rearrangement inequality for pairs of $n \times n$ matrices: Let $\|A\|_p$ denote $(\text{Tr}(A^*A)^{p/2})^{1/p}$, the C^p trace norm of an $n \times n$ matrix A . Consider the quantity $\|A+B\|_p^p + \|A-B\|_p^p$. Under certain positivity conditions, we show that this is nonincreasing for a natural “rearrangement” of the matrices A and B when $1 \leq p \leq 2$. We conjecture that this is true in general, without any restrictions on A and B . Were this the case, it would prove the analog of Hanner’s inequality for L^p function spaces, and would show that the unit ball in C^p has the exact same moduli of smoothness and convexity as does the unit ball in L^p for all $1 < p < \infty$. At present this is known to be the case only for $1 < p \leq 4/3$, $p = 2$, and $p \geq 4$. Several other rearrangement inequalities that are of interest in their own right are proved as the lemmas used in proving the main results.

1 Introduction

We prove certain rearrangement inequalities for matrices, the main results being Theorems 2.3 and 2.4 below. These rearrangement inequalities pertain to the non-commutative version [2, 9] of Hanner’s inequalities [5].

Hanner’s inequalities for any L^p function space state that [7, Theorem 2.5]

$$\|f + g\|_p^p + \|f - g\|_p^p \geq (\|f\|_p + \|g\|_p)^p + \left| \|f\|_p - \|g\|_p \right|^p \quad (1.1)$$

for $1 \leq p \leq 2$. The inequality reverses for $2 \leq p \leq \infty$.

Now, specialize to the case of $L^p(\mathbb{R}^n)$ with Lebesgue measure, and let f and g be non negative functions on \mathbb{R}^n . Further, let f^* and g^* denote their respective spherical symmetric decreasing rearrangements. (See, e.g., [7] for definitions.) The Chiti–Tartar inequality states [7, Theorem 3.5] that for any $p \geq 1$

$$\|f - g\|_p \geq \|f^* - g^*\|_p, \quad (1.2)$$

It is also well known that

$$\|f + g\|_p \leq \|f^* + g^*\|_p. \quad (1.3)$$

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One can extend the notion of spherical symmetric decreasing rearrangement to complex valued functions simply by putting $f^* = |f|^*$.

To connect (1.2) and (1.3) to (1.1), consider the sum $\|f + g\|_p^p + \|f - g\|_p^p$. It is natural to ask how this compares with $\|f^* + g^*\|_p^p + \|f^* - g^*\|_p^p$. For $p = 2$, the answer is clear; both quantities reduce to $2(\|f\|_2^2 + \|g\|_2^2)$.

For other values of p , we have the following.

1.1 LEMMA. *For all $1 \leq p \leq 2$, and all complex-valued functions f and g in $L^p(\mathbb{R}^n)$*

$$\|f + g\|_p^p + \|f - g\|_p^p \geq \|f^* + g^*\|_p^p + \|f^* - g^*\|_p^p . \tag{1.4}$$

For $p > 2$, the inequality reverses.

Proof: For non negative f and g , this lemma follows from a theorem in [1, Theorem 2.2], which states that for non-negative f and g , $\int J(f(x), g(x))dx \geq \int J(f^*(x), g^*(x))dx$ if $\frac{\partial^2}{\partial x \partial y} J(x, y) \leq 0$. It is easy to check that, with $J(x, y) = |x + y|^p + |x - y|^p$, we have $\frac{\partial^2}{\partial x \partial y} J(x, y) \leq 0$ for $1 \leq p \leq 2$ and ≥ 0 for $2 \leq p < \infty$. This establishes (1.4) for non-negative functions.

To complete the proof for complex functions it clearly suffices to prove that if f, g are replaced by $|f|, |g|$ then the left side of (1.4) decreases (resp. increases) for $1 \leq p \leq 2$ (resp. $2 \leq p < \infty$). A pointwise inequality suffices for this. For any two real numbers a and b , and any $-1 \leq t \leq 1$, define the functions $c(t)$ by

$$c(t) = (a^2 + b^2 + 2abt)^{p/2} + (a^2 + b^2 - 2abt)^{p/2} . \tag{1.5}$$

This function is strictly concave for $p < 2$, and strictly convex for $p > 2$. Since $c'(0) = 0$ in either case, we have that $t = \pm 1$ minimizes c for $p < 2$, while $t = \pm 1$ maximizes c for $p > 2$. Taking $a = |f(x)|$, $b = |g(x)|$ and $t = (\overline{f(x)}g(x) + f(x)\overline{g(x)})/(2|f(x)||g(x)|)$, we see that for $p < 2$

$$|f(x) - g(x)|^p + |f(x) + g(x)|^p \geq (|f(x)| + |g(x)|)^p + ||f(x)| - |g(x)||^p , \tag{1.6}$$

and this provides the extension to the complex case. □

Thus, we have the following situation: Consider all complex-valued functions ϕ and γ such that $|\phi|, |\gamma|$ are equimeasurable with $|f|$ and $|g|$, respectively (and, therefore, have the same right side of (1.1) as the f, g pair). Which choice will minimize the left side of (1.1) when $1 \leq p \leq 2$ (and maximize it when $2 \leq p < \infty$)?

One answer is the pair f^* and g^* . It is not the only answer, however, since any further equimeasurable rearrangement such that the set of level sets of f and those of g are the same will minimize the left side of (1.1).

A converse to Lemma 1.1 is the following, which says that the parallelogram identity for $p = 2$ holds as an inequality for $p \neq 2$.

1.2 LEMMA. *For all $1 \leq p \leq 2$, and all complex-valued functions f and g in $L^p(\mathbb{R}^n)$*

$$\|f + g\|_p^p + \|f - g\|_p^p \leq \lim_{y \rightarrow \infty} (\|f + \tau_y g\|_p^p + \|f - \tau_y g\|_p^p) = 2\|f\|_p^p + 2\|g\|_p^p , \tag{1.7}$$

where τ_y is translation by $y \in \mathbb{R}^n$. For $p > 2$, the inequality reverses.

The proof follows from the fact that that the function c in (1.5) is maximized at $t = 0$ for $p < 2$, and minimized there for $p > 2$ by the same convexity argument.

Our goal here is to extend Lemmas 1.1 and 1.2 to matrices. This will first require some discussion and notation.

2 Definitions and Main Theorems

Let A be any $n \times n$ matrix. Then $|A| = \sqrt{A^*A}$ and, for $1 \leq p < \infty$, $\|A\|_p = (\text{Tr}|A|^p)^{1/p}$. The analogue of Hanner's inequality (1.1) is

$$\|A + B\|_p^p + \|A - B\|_p^p \geq (\|A\|_p + \|B\|_p)^p + \|\|A\|_p - \|B\|_p\|^p \tag{2.1}$$

for $1 \leq p \leq 2$. The inequality reverses for $2 \leq p \leq \infty$.

Inequality (2.1) was proved in [2] for the following cases:

1. For all $1 \leq p \leq 4/3$ and $4 \leq p \leq \infty$, and of course $p = 2$.
2. For all $1 \leq p \leq \infty$ if $A + B$ and $A - B$ are positive semidefinite.

We conjecture that (2.1) holds for all A, B . (In [2] the condition in item 2 was incorrectly stated for the case $2 \leq p \leq \infty$; we are grateful to C. King for pointing out this error.)

Let $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$ be the singular values of an $n \times n$ matrix A , i.e., the eigenvalues of $|A|$. Let $\Sigma_{\uparrow}(A)$ and $\Sigma_{\downarrow}(A)$ be the $n \times n$ matrices defined by

$$\Sigma_{\uparrow}(A) = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} \quad \text{and} \quad \Sigma_{\downarrow}(A) = \begin{bmatrix} \sigma_n & & & \\ & \sigma_{n-1} & & \\ & & \ddots & \\ & & & \sigma_1 \end{bmatrix}. \tag{2.2}$$

We note, for later use in the proof of Theorems 2.3 and 2.4, that if A and B are Hermitian and $A > |B|$ then

$$\Sigma_{\uparrow}(A) > \Sigma_{\uparrow}(B) \quad \text{and} \quad \Sigma_{\downarrow}(A) > \Sigma_{\downarrow}(B). \tag{2.3}$$

The rearrangement $\Sigma_{\downarrow}(A)$ is considered in the book of Horn and Johnson [6] in the same notation apart from the arrow. (They only consider the decreasing ordering of the singular values). In problem 18 in section 3.5 [6], a proof is sketched of the analog of (1.2) for matrices: Namely that

$$\|\Sigma_{\downarrow}(A) - \Sigma_{\downarrow}(B)\| \leq \|A - B\|$$

for any unitarily invariant norm, and hence for the C^p norms in particular. However, the methods employed there do not seem to be useful when the direction of the inequality depends on the particular unitarily invariant norm under consideration, as in the matrix analogs of Lemmas 1.1 and 1.2, which are the following conjectures:

2.1 Conjecture. For all $1 \leq p \leq 2$, and all complex-valued $n \times n$ matrices A and B

$$\|A + B\|_p^p + \|A - B\|_p^p \geq \|\Sigma_{\uparrow}(A) + \Sigma_{\uparrow}(B)\|_p^p + \|\Sigma_{\uparrow}(A) - \Sigma_{\uparrow}(B)\|_p^p. \tag{2.4}$$

For $p > 2$, the inequality reverses.

2.2 Conjecture. For all $1 \leq p \leq 2$, and all complex-valued $n \times n$ matrices A and B

$$\|A + B\|_p^p + \|A - B\|_p^p \leq \|\Sigma_{\uparrow}(A) + \Sigma_{\downarrow}(B)\|_p^p + \|\Sigma_{\uparrow}(A) - \Sigma_{\downarrow}(B)\|_p^p. \tag{2.5}$$

For $p > 2$, the inequality reverses.

Note that Conjecture 2.1, if true, implies (2.1) in full generality because it reduces the matrix case to the commutative case, Theorem 1.1, namely to diagonal matrices (which are just functions on $\{1, 2, \dots, n\}$). We also note that (2.4) holds with the reverse inequality for p an even integer and also without restriction on A and B . This is some evidence for the validity of the conjecture.

We can prove the following cases (where $X \geq Y$ means that $X - Y$ is positive-semidefinite).

2.3 THEOREM. *Conjecture 2.1 is true for $1 \leq p \leq 2$ if A and B are self adjoint and $A \geq B \geq 0$.*

2.4 THEOREM. *Conjecture 2.2 is true for $1 \leq p \leq 2$ if A and B are self adjoint and $A \geq |B|$.*

While both theorems contain positivity conditions, at least Theorem 2.4 does not require B to be positive. The presence of the positivity conditions in Theorems 2.3 and 2.4 reflects an important difference between the cases of matrices and functions. In the case of functions, the simple pointwise inequality (1.6) sufficed to reduce matters to the consideration of positive functions. In the case of matrices, this is not possible: There is actually an inequality that goes in the direction *opposite* to (1.6).

2.5 LEMMA. *Let A and B be self adjoint $n \times n$ matrices, and suppose that $A \geq |B| \geq 0$. Then for $1 \leq p \leq 2$,*

$$\mathrm{Tr}((A + B)^p + (A - B)^p) \leq \mathrm{Tr}((A + |B|)^p + (A - |B|)^p) .$$

Proof: Let X denote the positive part of B , and let Y denote the negative part so that $B = X - Y$ and $|B| = X + Y$. Define functions $f(t)$ and $g(t)$ for $0 \leq t \leq 1$ by

$$f(t) = \frac{1}{p} \mathrm{Tr}((A + tB)^p + (A - tB)^p) \quad \text{and} \quad g(t) = \frac{1}{p} \mathrm{Tr}((A + t|B|)^p + (A - t|B|)^p) .$$

Clearly $f(0) = g(0)$, We claim that for each t with $0 < t \leq 1$, $f'(t) > g'(t)$. To see carry out the computation that demonstrates this, define the positive semidefinite matrices

$$Z_1 = A + t(X + Y) \quad Z_2 = A + t(X - Y) \quad Z_3 = A - t(X - Y) \quad \text{and} \quad Z_4 = A - t(X + Y) .$$

Then

$$\begin{aligned} f'(t) - g'(t) &= \\ &= \mathrm{Tr} \left((X + Y)(Z_1^{p-1} - Z_4^{p-1}) \right) - \mathrm{Tr} \left((X - Y)(Z_2^{p-1} - Z_3^{p-1}) \right) = \\ &= \mathrm{Tr} \left(X(Z_1^{p-1} + Z_3^{p-1} - Z_2^{p-1} - Z_4^{p-1}) \right) + \mathrm{Tr} \left(Y(Z_1^{p-1} + Z_2^{p-1} - Z_3^{p-1} - Z_4^{p-1}) \right) = \\ &= \mathrm{Tr} \left(X([Z_1^{p-1} - Z_2^{p-1}] + [Z_3^{p-1} - Z_4^{p-1}]) \right) + \mathrm{Tr} \left(Y([Z_1^{p-1} - Z_3^{p-1}] + [Z_2^{p-1} - Z_4^{p-1}]) \right) . \end{aligned}$$

Because $0 \leq p-1 \leq 1$, the operator monotonicity of $(p-1)$ st powers implies that all of the differences in square brackets are positive. Hence $f'(t) - g'(t) \geq 0$, and so $f(1) \geq g(1)$. \square

3 Proof of Theorems 2.3 and 2.4

The proofs of both theorems rely on a rearrangement inequality for alternating products of two positive $n \times n$ matrices A and B . The fact needed in the proofs is that, for integer $s > 0$, the quantity

$$\mathrm{Tr}(BABA \dots BAB) = \mathrm{Tr}(B(B^{1/2}AB^{1/2})^s) \tag{3.1}$$

is nonincreasing if we rearrange A and B oppositely, and nondecreasing if we rearrange A and B similarly. That is, the quantity in (3.1) does not increase if we replace A by $\Sigma_{\uparrow}(A)$ and B by $\Sigma_{\downarrow}(B)$, and does not decrease if replace A by $\Sigma_{\uparrow}(A)$ and B by $\Sigma_{\uparrow}(B)$. The following theorems assert this, and somewhat more.

3.1 THEOREM. *For any two positive-semidefinite $n \times n$ matrices A and B , any numbers $r \geq 0$ and $s \geq 1$,*

$$\mathrm{Tr} \left(B^r (B^{1/2} A B^{1/2})^s \right) \geq \mathrm{Tr} \left((\Sigma_{\uparrow}(A))^s (\Sigma_{\downarrow}(B))^{s+r} \right) . \quad (3.2)$$

3.2 THEOREM. *For any two positive-semidefinite $n \times n$ matrices A and B , any number $r \geq 0$ and any integer $s \geq 1$,*

$$\mathrm{Tr} \left((\Sigma_{\uparrow}(A))^s (\Sigma_{\uparrow}(B))^{s+r} \right) \geq \mathrm{Tr} \left(B^r (B^{1/2} A B^{1/2})^s \right) . \quad (3.3)$$

Unlike Theorem 3.1, Theorem 3.2 requires that s be an integer. This condition on s would be unnecessary if a natural generalization of an inequality of Lieb and Thirring [8, Appendix B] were established, as we explain in an appendix to this paper, where further trace inequalities are conjectured and proved. These are closely related to results and a conjecture in our earlier paper [3].

One tool used in the proof of Theorems 3.1 and 3.2 is a ‘‘layer cake representation’’ for positive matrices. Let C be any positive $n \times n$ matrix with spectral decomposition $C = \sum_{i=1}^s \lambda_i u_i u_i^*$ in which the eigenvalues λ_j are arranged in decreasing order. Let

$$P_j = \sum_{i=1}^j u_i u_i^* . \quad (3.4)$$

Then P_j is the orthogonal projection onto an eigenspace corresponding to the j largest eigenvalues of C , and clearly $P_j \subset P_{j+1}$. With $P_0 = 0$, we have

$$C = \sum_{j=1}^n \lambda_j (P_j - P_{j-1}) = \lambda_n P_n + \sum_{j=1}^{n-1} (\lambda_j - \lambda_{j+1}) P_j .$$

Define $c_j = \lambda_j - \lambda_{j+1}$ for $1 \leq j \leq n-1$ and $c_n = \lambda_n$. Then

$$C = \sum_{j=1}^n c_j P_j . \quad (3.5)$$

Note that each c_j is non negative, and $\sum_{j=1}^n c_j = \lambda_1$. Therefore, if $\|C\|_{\infty} = 1$, i.e., $\lambda_1 = 1$, then (3.5) presents C as a convex combination of projections.

Proof of Theorem 3.1: *(Step One: Reduction to the case $r = 0$)* Observe that, with $X = B^{1/2} A B^{1/2}$ and $Y = B^{r/s}$, $\mathrm{Tr}(B^r (B^{1/2} A B^{1/2})^s) = \mathrm{Tr}(X^s Y^s)$. The inequality of [8, Appendix B] asserts that $\mathrm{Tr}(X^s Y^s) \geq \mathrm{Tr}((Y^{1/2} X Y^{1/2})^s)$. Therefore,

$$\mathrm{Tr}(B^r (AB)^s) \geq \mathrm{Tr}((B^{(s+r)/2s} A B^{(s+r)/2s})^s) .$$

Given the validity of (3.2) in the case $r = 0$, we have

$$\begin{aligned} \mathrm{Tr}((B^{(s+r)/2s} A B^{(s+r)/2s})^s) &\geq \mathrm{Tr} \left((\Sigma_{\downarrow}(B)^{(s+r)/2s})^s (\Sigma_{\uparrow}(A))^s (\Sigma_{\downarrow}(B)^{(s+r)/2s})^s \right) \\ &= \mathrm{Tr} \left((\Sigma_{\uparrow}(A))^s (\Sigma_{\downarrow}(B))^{s+r} \right) . \end{aligned} \quad (3.6)$$

(Step Two: Proof of (3.2) for $r = 0$) This is based on Epstein's concavity theorem [4], and the layer cake representation (3.5). Without loss of generality, we may suppose that $\|A\| = \|B\| = 1$.

Let $C = A^s$, and note that $\|C\| = 1$ as well. Then

$$\mathrm{Tr}(B^{1/2}AB^{1/2})^s = \mathrm{Tr}(B^{1/2}C^{1/s}B^{1/2})^s .$$

By Epstein's theorem, $\mathrm{Tr}(B^{1/2}C^{1/s}B^{1/2})^s$ is a concave function of C . Since $\|C\| = 1$, the layer cake representation $C = \sum_{j=1}^n c_j P_j$ is a convex combination of projections, and hence

$$\mathrm{Tr}(B^{1/2}C^{1/s}B^{1/2})^s \geq \sum_{j=1}^n c_j \mathrm{Tr}(B^{1/2}P_j^{1/s}B^{1/2})^s \quad (3.7)$$

Next, since each P_j is an orthogonal projection, $P_j^{1/s} = P_j = P_j^{1/2}$. Also, for any two positive semidefinite matrices X and Y , $\mathrm{Tr}((X^{1/2}YX^{1/2})^s) = \mathrm{Tr}((Y^{1/2}XY^{1/2})^s)$, since, in the positive definite case, $X^{1/2}YX^{1/2}$ and $Y^{1/2}XY^{1/2}$ are similar matrices. Hence (3.7) becomes

$$\mathrm{Tr}(B^{1/2}C^{1/s}B^{1/2})^s \geq \sum_{j=1}^n c_j \mathrm{Tr}(P_j B P_j)^s .$$

Now do the same thing for B : Let $D = B^s$, and let $D = \sum_{k=1}^n d_k Q_k$ be the layer-cake representation of D . Again, since the largest eigenvalue of D is 1, this displays D as a convex combination of projections. Again applying Epstein's theorem, and using the fact that each $Q_k^{1/s} = Q_k$, we deduce that for each j ,

$$\mathrm{Tr}(P_j B P_j)^s = \mathrm{Tr}(P_j D^{1/s} P_j)^s \geq \sum_{k=1}^n d_k \mathrm{Tr}(P_j Q_k P_j)^s = \sum_{k=1}^n d_k \mathrm{Tr}(Q_k P_j Q_k)^s . \quad (3.8)$$

Now, $Q_k P_j Q_k$ is a positive contraction. A vector v is an eigenvector of this contraction with eigenvalue 1 if and only if v belongs to the images of both P_j and Q_k . Let $r_{j,k}$ be the dimension of the intersection of the images of both P_j and Q_k . This is the geometric multiplicity of 1 as an eigenvalue of $Q_k P_j Q_k$. Since all of the other eigenvalues are non negative,

$$\mathrm{Tr}(Q_k P_j Q_k)^s \geq r_{j,k}$$

for all s . (In fact, it converges to this value as s increases).

For any two subspaces \mathcal{V}_1 and \mathcal{V}_2 ,

$$\dim(\mathcal{V}_1 + \mathcal{V}_2) + \dim(\mathcal{V}_1 \cap \mathcal{V}_2) = \dim(\mathcal{V}_1) + \dim(\mathcal{V}_2) ,$$

and we have

$$r_{j,k} \geq \max\{ \mathrm{Tr}(P_j) + \mathrm{Tr}(Q_k) - n , 0 \} .$$

There is equality in this inequality if we replace P_j by $\Sigma_{\uparrow}(P_j)$, and Q_k by $\Sigma_{\downarrow}(Q_k)$. Hence, since these matrices commute,

$$\begin{aligned} \mathrm{Tr}(AB)^s &\geq \sum_{j,k=1}^n c_j d_k \mathrm{Tr}((\Sigma_{\uparrow}(P_j)\Sigma_{\downarrow}(Q_k))^s) = \mathrm{Tr} \left(\left(\sum_{j=1}^n c_j \Sigma_{\uparrow}(P_j) \right) \left(\sum_{k=1}^n d_k \Sigma_{\downarrow}(Q_k) \right) \right) \\ &= \mathrm{Tr}(\Sigma_{\uparrow}(A^s)\Sigma_{\downarrow}(B^s)) = \mathrm{Tr}((\Sigma_{\uparrow}(A)\Sigma_{\downarrow}(B))^s) . \end{aligned} \quad (3.9)$$

□

Proof of Theorem 3.2: Let N denote the integral value of s . Notice that $\text{Tr}(B^r(B^{1/2}AB^{1/2})^N) = \text{Tr}(B^r(AB)^N)$. Expand A , B and B^r in their layer cake representations as above. Taking the trace, we get a linear combination with positive coefficients of terms such as $\prod_{j=1}^{2N+1} P_j$, where each P_j is a projection coming from one of the layer cake expansions. By cyclicity of the trace, we may assume that $\text{Tr}(P_1) = \min \{ \text{Tr}(P_j) : 1 \leq j \leq 2N + 1 \}$. Then since $\prod_{j=2}^{2N+1} P_j$ is a contraction, if we compute

$\text{Tr} \left(\prod_{j=1}^{2N+1} P_j \right)$ in a basis of eigenvectors of P_1 , we certainly find that

$$\left| \text{Tr} \left(\prod_{j=1}^{2N+1} P_j \right) \right| \leq \text{Tr}(P_1) = \min \{ \text{Tr}(P_j) : 1 \leq j \leq 2N + 1 \} .$$

There is equality in case the P_j all commute, and are all “nested” which is what happens if we replace A and B by $\Sigma_{\uparrow}(A)$ and $\Sigma_{\uparrow}(B)$ respectively, or just as well, by $\Sigma_{\downarrow}(A)$ and $\Sigma_{\downarrow}(B)$ respectively. \square

Proof of Theorem 2.3: For any positive matrix C and any p with $1 < p < 2$,

$$C^p = k_p C \int_0^\infty \left(\frac{1}{t} - \frac{1}{t+C} \right) t^{p-1} dt = k_p \int_0^\infty \left(\frac{C}{t^2} - \frac{1}{t} + \frac{1}{t+C} \right) t^p dt , \quad (3.10)$$

where $k_p > 0$ is a normalization constant. We alternately set $C = A + B > 0$ and $C = A - B > 0$ in the integrand of (3.10) and take the trace. Since $\text{Tr}(A) = \text{Tr}(\Sigma_{\uparrow}(A))$ and, by (2.3), $\Sigma_{\uparrow}(A) \pm \Sigma_{\uparrow}(B) > 0$, it suffices, for our proof, to show that for each $t > 0$

$$\text{Tr} \left((t + A + B)^{-1} + (t + A - B)^{-1} \right) \geq \text{Tr} \left((t + \Sigma_{\uparrow}(A) + \Sigma_{\uparrow}(B))^{-1} + (t + \Sigma_{\uparrow}(A) - \Sigma_{\uparrow}(B))^{-1} \right) \quad (3.11)$$

Let H denote $A + t$. By (2.3) we have that $K := H^{-1/2}BH^{-1/2}$ satisfies $0 < K < 1$. Therefore, it is legitimate to expand

$$(H \pm B)^{-1} = H^{-1/2}(1 \pm K)^{-1}H^{-1/2} = H^{-1/2} \sum_{j=0}^{\infty} (-1)^j (\pm K)^j H^{-1/2} . \quad (3.12)$$

If these two expressions, \pm , are added, the left side of (3.11) becomes

$$2 \sum_{j=0}^{\infty} \text{Tr}(H^{-1}K^{2j}) = 2 \sum_{j=0}^{\infty} \text{Tr}H^{-1}(BH^{-1})^{2j} . \quad (3.13)$$

An expression similar to this is obtained for the right side of (3.11), except that B is replaced by $\Sigma_{\uparrow}(B)$ and H^{-1} is replaced by $\Sigma_{\downarrow}(H^{-1})$, which arises from the fact that $(t + \Sigma_{\uparrow}(A))^{-1} = \Sigma_{\downarrow}((t + A)^{-1})$. By Theorem (3.1), this replacement cannot increase each term in (3.13). \square

Proof of Theorem 2.4: By Lemma 2.5, we may replace B by $|B|$, and since $A \geq |B|$, both $A + |B|$ and $A - |B|$ are non negative. Hence, the integral representation used in the proof of Theorem 2.3 may be applied. Instead of rearranging oppositely and applying Theorem 3.1, rearrange similarly, and apply Theorem 3.2. Using one additional but obvious fact – that $\Sigma_{\uparrow}(B) = \Sigma_{\uparrow}(|B|)$ – the theorem is proved. \square

A Appendix: Remarks on Theorem 3.2

We have made use of the inequality

$$\mathrm{Tr}(Y^{1/2}XY^{1/2})^s \leq \mathrm{Tr}(X^sY^s), \quad (\text{A.1})$$

valid for all positive semidefinite $n \times n$ matrices X and Y and all $s \geq 1$. This inequality was proved in [8, Appendix B] using Epstein's theorem [4], which asserts the concavity for all $s \geq 1$ of the function f_s , given by

$$f_s(A) = \mathrm{Tr}((B^{1/2}A^{1/s}B^{1/2})^s) \quad (\text{A.2})$$

on the set of positive semidefinite $n \times n$ matrices, where B is some fixed positive semidefinite $n \times n$ matrix. In [3], we conjectured that for $1/2 < s < 1$, f_s is convex. Indeed, $f_{1/2}$ is convex, since $\mathrm{Tr}((B^{1/2}A^2B^{1/2})^{1/2}) = \|AB^{1/2}\|_1$. We used the concavity of f_s for $s \geq 1$ to prove a Minkowski type inequality for traces, and showed how this yielded a proof of the strong subadditivity of the quantum mechanical entropy. The conjectured convexity would have done the same thing.

Were f_s convex for $1/2 < s < 1$, the kind of proof in [8] of (A.1) would carry over to a proof that for all positive semidefinite $n \times n$ matrices A and B , and all such s ,

$$\mathrm{Tr}(A^sB^s) \leq \mathrm{Tr}((B^{1/2}AB^{1/2})^s) \quad (\text{A.3})$$

Using the convexity of $f_{1/2}$ proved above, we now prove (A.3) for $s = 1/2$. Let $C = A^{1/2}$, and introduce $g(C)$ and $h(C)$ where $g(C) = \mathrm{Tr}(CB^{1/2})$ and $h(C) = \mathrm{Tr}(B^{1/2}C^2B^{1/2})^{1/2}$. By what has been said above, $h(C) - g(C)$ is a convex function of C . Let $C^{(d)}$ denote the part of C that is diagonal in a basis that diagonalizes B , and let $C^{(o)}$ denote the off diagonal part. As in [8], one sees that

$$k(t) = h(C^{(d)} + tC^{(o)}) - g(C^{(d)} + tC^{(o)})$$

satisfies $k(0) = k'(0) = 0$, and since it is convex, $k(1) \geq 0$. This proves (A.3) for $s = 1/2$. Given the conjectured convexity of f_s for $1/2 < s < 1$, this argument would establish (A.3) for $1/2 < s < 1$.

Another possible generalization of (A.1) is the following:

$$\mathrm{Tr}(Y^r(Y^{1/2}XY^{1/2})^s) \leq \mathrm{Tr}(X^sY^{s+r}) \quad (\text{A.4})$$

for all positive semidefinite $n \times n$ matrices X and Y , all $r > 0$, and all $s \geq 1$. Were this true, one could easily remove the restriction that s be an integer in Theorem 3.2. This would be true if it were the case that

$$\tilde{f}_s(A) = \mathrm{Tr}(B^r(B^{1/2}A^{1/s}B^{1/2})^s)$$

defined a concave function of the positive semidefinite matrix A for $s \geq 1$, and B a given positive semidefinite matrix.

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