A User's Guide to the Conley-Morse Database

Hiroshi Kokubu, Kelly Spendlove

Abstract

The past few decades of dynamical systems theory have established that multiparameter nonlinear dynamical systems can exhibit extremely complex behavior with respect to both the system variables and parameters. Such complex behavior proven in theoretical work has to be contrasted with the capabilities of application; in the case of modeling multiscale processes, for instance, measurements may be of limited precision, parameters are rarely known exactly and nonlinearities are often not derived from first principles. This contrast suggests that extracting robust features which persist over a range of parameter values is of greater importance than a detailed understanding of the fine structure at some particular parameter. That is, *the resolution at which one analyzes the problem is of fundamental importance*. The goal of this paper is to present an overview of an approach to dynamics which accounts for the role of resolution. This approach is used to obtain a coarse yet robust description of the global dynamics at a resolution specified *a priori*. A crude but rigorous characterization of the local dynamics is given via the Conley Index - an algebraic topological invariant. Foremost, we wish to convey these ideas to a general audience, casting the theory in a simple combinatorial framework to provide what one needs to know to become a 'user'.

1 Introduction

Countless questions from a variety of the sciences lead one to attempt to analyze the global dynamics of multiparameter dynamical systems. We will see just a few examples in these notes. However, when shouldering the task of studying global dynamics we must acknowledge some inherent obstacles which have been established through the work of the dynamical systems community in the last few decades. The primary obstacle is that global dynamical structures can vary on all scales in both phase space and parameter space. For instance *chaos*, a phenomenon of deterministic systems made famous by science popularizers, implies a sensitivity to initial conditions.[1] That is, arbitrarily small perturbations or errors may lead to order one differences in the behavior of trajectories. Thus in a chaotic system the behavior of an individual orbit obtained through a numerical simulation may not be expected to accurately represent the true orbit of the system.

Observation of the complications due to chaos dates back to Henri Poincaré around the turn of the 19th century [2]. The subsequent classical dynamical systems theory handles this by focusing on the existence and structure of invariant sets in contrast to the behavior of a particular orbit. However, a similar phenomenon presents itself in parameter space: there exist systems where an infinitesimal perturbation to the parameters can change the global dynamics [12]. This implies if the parameters are not known precisely then even a perfect simulation (not even speaking of the difficulties involved in numerical analysis) may suggest the wrong dynamic.

Such obstacles inherent to the classical approach to dynamics must be contrasted with the capabilities of application. Models of multiscale processes, especially in the biological sciences, are rarely derived from first principles, but instead through a series of approximations or heuristics. Furthermore, parameter values are often known only to an order of magnitude or even unknown

---

[1]For a historical account of *chaos* see [2, 8]; a mathematical treatment can be found in [12, 14].

all together. Finally, computations may contain errors of various orders of magnitude. Thus the exact values produced by a model at particular parameter values cannot be expected to, and should not intend to, match those of the physical system. Furthermore, these results are often compared against experimental data which itself is of limited precision and often noisy. Therefore, in the context of application the possibility of interpreting an incorrect dynamic is even more likely.

Such a contrast suggests the value of reconsidering the resolution of our analysis. That is, extracting coarser and more robust descriptions of the dynamics is more important than understanding the fine structure of a model at a particular parameter. Furthermore, resolution could be a subject of study in itself and a mathematical theory attempting to robustly describe global dynamics should be capable at any fixed *a priori* resolution. In this survey we will review computational methods used to study the global dynamics of nonlinear multiparameter systems which are rooted in the ideas of C. Conley [5]. In line with the obstacles presented above, the fundamental objects of the theory are coarse, yet this allows them to be robust with respect to perturbations. At its core the theory consists of two components - *decomposition* and *reconstruction*; first decomposing the dynamics into gradient-like, i.e. strictly non-recurrent, and recurrent parts and using this representation to infer the global dynamics, then 'reconstructing' or characterizing the behavior of the recurrent dynamics produced from the decomposition.

2   Organization and Philosophy

This survey will introduce the Conley Theory in an unorthodox manner, beginning with an introduction to combinatorial dynamics. Graphs are by now a ubiquitous mathematical idea; an appropriate starting point to introduce the theory and related structures. We refer to studying dynamics on graphs as 'combinatorial dynamics'. We will then proceed toward how to translate more traditional dynamical models, such as continuous functions, or *maps*, and differential equations into combinatorial dynamical models.

Throughout the survey we will introduce ideas or examples in highlighted boxes. These will be differentiated into *Example* and *Process*. Example is typically an application of our ideas, while Process is a core algorithm or procedure of the Conley Theory.

For a technical mathematical introduction to the database, we suggest [1, 4]; excellent surveys of the Conley Theory can be found in [9, 10, 13]. The aim of this survey is to be nontechnical, and attempt to illustrate ideas or theorems with an example rather than definition or proof. We relegate mathematical definitions to Appendices A–C.

The second aim of this survey is to introduce the reader to mathematical terminology and jargon, especially from the field of dynamical systems and Conley Theory; words such as robust, coarse, global, local, et cetera (many of these words appeared in the introduction). When such words appear we will make an effort to introduce the intuition behind their usage.

3   Combinatorial Dynamics

Much of modern data lends itself naturally toward a combinatorial representation - that is, through undirected and directed graphs. In this survey we will focus on directed graphs. Intuitively speaking, as our motivation lies in studying the dynamics of these objects, most often we view the vertices as states of a system and the edges correspond to the behavior of the system, i.e. the 'dynamics'.

A simple example is a 'contact network': a graph-based model for the spread of infectious diseases through considering person-to-person contacts within a community, as in Example 1.

One of the most relevant questions when analyzing such directed graphs is how vertices tend to group together into clusters. From a dynamical point of view, this corresponds to states of the system that are recurrent in some fashion. Here, a state or set of states is recurrent in the sense

Example 1: Contact Network

Each vertex corresponds to a person in the community, with an edge representing if two people are in contact. Further, contacts may be asymmetric, representing a person more likely to infect one of their contacts than to become infected by that contact.

Figure 1 depicts a highly asymmetric contact network; perhaps vertex F represents a healthy care worker, G and H as family members, and A through E patients.
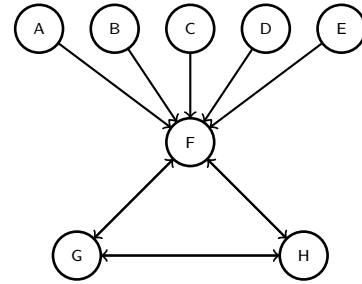

Figure 1: Contact Network

that it may be visited infinitely many times in the future. For instance, in the graph depicted in Figure 1 the cycle of vertices F − G − H represents a recurrent set.

In this section we address a natural decomposition of the directed graph into such clusters. However, the precise definition of a cluster is ill-defined; largely dependent upon the intended application. For our purposes, we define a cluster as a concept similar to a strongly connected component - roughly, a subset of vertices in which for any two vertices of the set v_1 and v_2, there are directed paths from $v_1 \to v_2$ and $v_2 \to v_1$. The decomposition of a directed graph into its strongly connected components seems a ubiquitous concept [6]. There are three fundamental reasons for our focus on such components:

B1 The set of strongly connected components characterizes the 'global' behavior of the graph.

B2 The set of strongly connected components is efficiently computable.

B3 Examining strongly connected components lends itself to extracting rigorous results when considering directed graphs obtained from continuous processes.

3.1 Preliminaries

As introduced above, a directed graph G is *strongly connected* if there is a path from each vertex in the graph to any other vertex. The *strongly connected components* of G are its maximal strongly connected subgraphs.
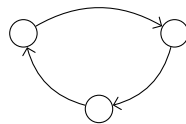

Figure 2: Strongly Connected Directed Graph

This definition implies that one vertex which does not lie on any cycle is then a strongly connected component (i.e. even if the vertex does not have a directed edge to itself). For instance, in the figure below, each vertex is a strongly connected component.

To get rid of such cases, we modify the definition to the set of *strongly connected path component* (SCPC): the maximal subset of vertices such that for each v there is a nontrivial path back to v.

Figure 3: An acylic directed graph where each vertex is a strongly connected component

3.2 Strongly Connected Path Components

One of the most important results regarding the set of SCPCs is that it has a natural order induced by the edges in directed graph:

For a graph $G = (V, E)$, let P be an index set for the collection of SCPCs. For $p, q \in \mathsf{P}$, define $p \leq q$ if there is a path in G starting from a vertex in q and ending at a vertex in p. It is in this sense the set of SCPCs is ordered.

Definition 3.1. Let $G = (V, E)$ be a directed graph. In the terminology of Conley Theory, the set of SCPCs is called a *combinatorial Morse Decomposition* of G. An individual strongly connected path component $\mathcal{M}(q)$ is called a *combinatorial Morse set*.

Addressing **B1**, it can be seen that the set of strongly connected components accounts for all of the recurrent behavior in the graph; any recurrent behavior must be contained in some strongly connected component.

In many applications the most important part of the graph is the maximal invariant set. An *invariant set* of vertices is a set $S \subset V$ such that if you let S move to all of the vertices it has an edge connecting to, the resulting set of vertices is again S. The largest such set of invariant sets is the maximal invariant set. A formal definition is provided in Appendix B. For instance, the maximal invariant set of the graph in Figure 2 is the graph itself; in Figure 3 there is no maximal invariant set (that is, it is the empty set).

If v belongs to the maximal invariant set S then there must exist two SCPCs, q and p with $q > p$, such that there is a directed path from q to v and a directed path from v to p. Therefore any vertex lying out of the set of SCPCs yet inside the maximal invariant set S lies on *gradient-like* dynamics between SCPCs. The dynamics are 'gradient-like' in the sense that they are not recurrent. It is precisely in this sense that the set of strongly connected components characterize the *global dynamics* of the graph, for this reason we refer to the dynamics inside of an SCPC as 'local dynamics' and off of an SCPC as 'global dynamics'.[2]

Turning toward **B2**, there are known linear time algorithms for computing the set of strongly connected components [6]. **B3** requires a bit more mathematical machinery, and will be discussed in Section 5.2.

3.3 Morse Graphs

Any directed graph $G = (V, E)$ is naturally decomposed into the set of strongly connected path components P with partial order $\leq$.[3] From this one can construct a new graph by collapsing each strongly connected path component to a single vertex and forming an edge $q \to p$ if $p \leq q$. This is entitled a *Morse graph* and denoted $\mathsf{MG}(G)$. The construction is further depicted in Process 1.

---

[2] One may be wondering about the vertices which do not satisfy the condition belonging to the maximal invariant set. These are considered fairly uninteresting from the dynamical point of view, as there they are not recurrent and are not contained between any two recurrent components.

[3] For the definition of *partial order* see Definition B.6 in Appendix B.

Process 1: Creating a Morse Graph
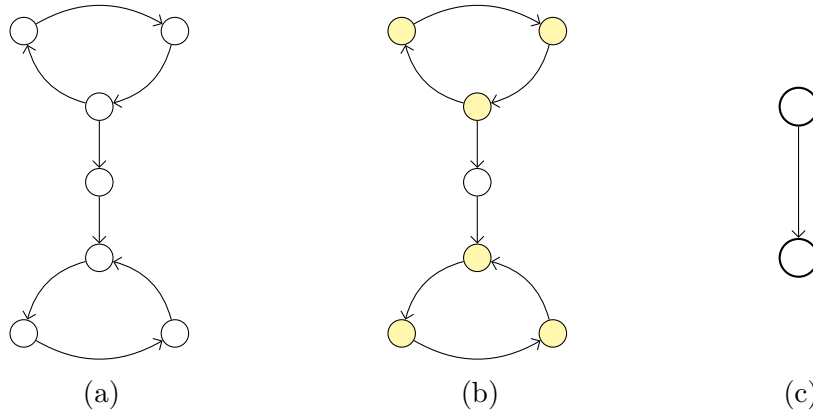

(a)   (b)   (c)

Figure 4: Creating a Morse Graph: (a) $G = (V, E)$; (b) Compute SCPCs; (c) Collapse each SCPC to its own vertex, create edges based on partial order to obtain $\mathsf{MG}(G)$. Simply put, a Morse graph is a compact, efficiently computable representation of the global dynamics.

Example 2: Sampled Social Network

Suppose we are interested in the propagation of information across a social network such as Twitter. For our system the states correspond to which users are in possession of the information. In a basic setting we may consider the following model: we have n users which correspond to n vertices; if user i is a follower of user j then there is a directed edge $j \rightarrow i$.

This graphs contains four strongly connected path components. Only one vertex does not lie in any component. While information may be propagated amongst users in a strongly connected path component, it can only be propagated further to other users according to the partial order on the SCPCs. In other words, off of the SCPCs the information moves in a gradient-like fashion. This idea is characterized by the Morse Graph of the network.
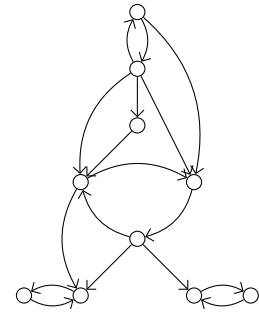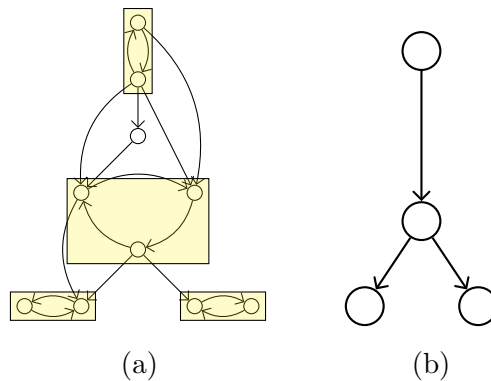

Figure 5: Sampled Network

(a)   (b)

Figure 6: (a) Capturing strongly connected components; (b) Associated Morse Graph

4 Combinatorialization: Discretizing a Dynamical System

Much of traditional dynamical systems theory is directed toward describing the properties of systems whose states lie on a continuum, for instance the real line $\mathbb{R}$. The abstract space comprised of the states of the system is entitled the *phase space*. For further information the reader should consult [14] for a brief introduction; such systems will also be discussed in briefly in later sections.

However, in this section we will describe a method of passing from the action of a system on a continuous space ($\mathbb{R}$, for instance) to a representation via a directed graph. To keep prerequisites to a minimum in this section, we will consider systems whose phase space X are subsets of Euclidean space, i.e. $X \subset \mathbb{R}^n$.

4.1 Discretizing Phase and Parameter Space

The fundamental construction to introduce is that of a *grid*. The construction depends upon some topological notions, for which the reader may consult [11, 14]. To provide some quick intuition, the topological notions of closure and interior correspond to whether one includes the 'boundary' points (closure) or removes 'boundary' points (interior) (see Definitions B.1–B.3 in Appendix B). We use $\mathrm{cl}(\cdot)$ and $\mathrm{int}(\cdot)$ to denote the closure and interior, respectively. With these heuristic definitions given, a grid is a finite collection $\mathcal{X}$ of nonempty, closed and bounded subsets of X with the following properties:

I. The union of the pieces is the whole space, i.e. $X = \cup_{\xi \in \mathcal{X}} \xi$

II. The pieces are in some sense not pathological, i.e. $\xi = \mathrm{cl}(\mathrm{int}(\xi))$ for all $\xi \in \mathcal{X}$

III. The pieces fit together nicely - not overlapping except on the boundary, i.e. $\xi \cap \mathrm{int}(\xi') = \emptyset$ for all $\xi \neq \xi'$

An example can be seen in Process 2.

4.2 Discretizing Dynamics

To discretize the dynamics one makes use of a *combinatorial multivalued map* $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$, which assigns to each element ξ of grid $\mathcal{X}$ a subset (possibly empty) $\mathcal{F}(\xi)$ of $\mathcal{X}$. Notice that a grid element ξ is a subset of the phase space, and thus can be thought to represent a collection of states of the system. Going a step further, they can be thought to correspond to vertices in a directed graph. Then the image ξ under $\mathcal{F}$ is naturally represented by directed edges from ξ to the collection of other vertices in $\mathcal{F}(\xi)$. Thus we can think of the combinatorial multivalued map as being equivalent to a directed graph, and use previously mentioned graph theoretic constructions to analyze the multivalued maps. The basic construction of $\mathcal{F}$ given $\mathcal{X}$ is shown in Process 2.

As one can choose the size of the grid elements to use, the concept of a grid allows one to study the dynamical system at a fixed *a priori* resolution. Beyond its importance with respect to the obstacles of the traditional theory, resolution has a larger role to play within the computational Conley Theory. In general one expects that for models of well understood phenomena or physical systems for which accurate measurements can be made it is appropriate to choose finer grids. In contrast, given a crude or heuristic model or a system for which only coarse measurements can be made, one expects that the results associated with cruder grids will provide more meaningful information.

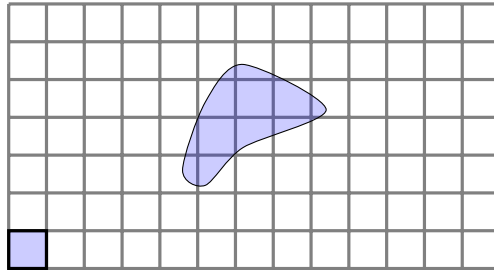1. For each $\xi \in \mathcal{X}$, consider $f(\xi)$.

Figure 7: ξ and its image $f(|\xi|)$.

2. Cover $f(\xi)$ with elements of the grid, and define this to be $\mathcal{F}(\xi)$. More precisely $\mathcal{F}(\xi) := \{\xi' \mid \xi' \cap f(\xi) \neq \emptyset\}$.
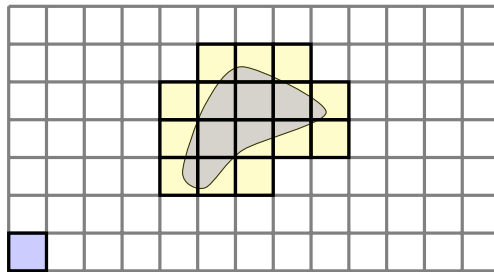

Figure 8: $f(\xi)$ covered by elements of $\mathcal{X}$ with nonempty intersection; this defines $\mathcal{F}(\xi)$

3. A combinatorial multivalued map $\mathcal{F}$ is also thought of as a directed graph: ξ is a vertex and $\mathcal{F}(\xi)$ describes the edges
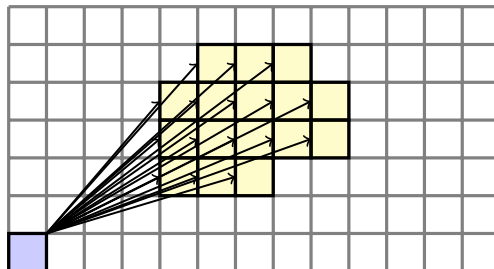

Figure 9: $\mathcal{F}$ as a directed graph

Example 3: Logistic Map

We highlight an example of translating a continuous nonlinear structure into a combinatorial multivalued map; the process we refer to as 'combinatorialization'. A natural place to begin is with the well-known logistic map:

$$f(x) = rx(1-x)$$

In particular we will consider an example given in [3], with $r = 2.5$. Figure 11 shows the graph of $f(x)$ in addition to the directed graph obtained by dividing the domain $X = [0, \frac{5}{8}]$ into five subintervals of equal length.
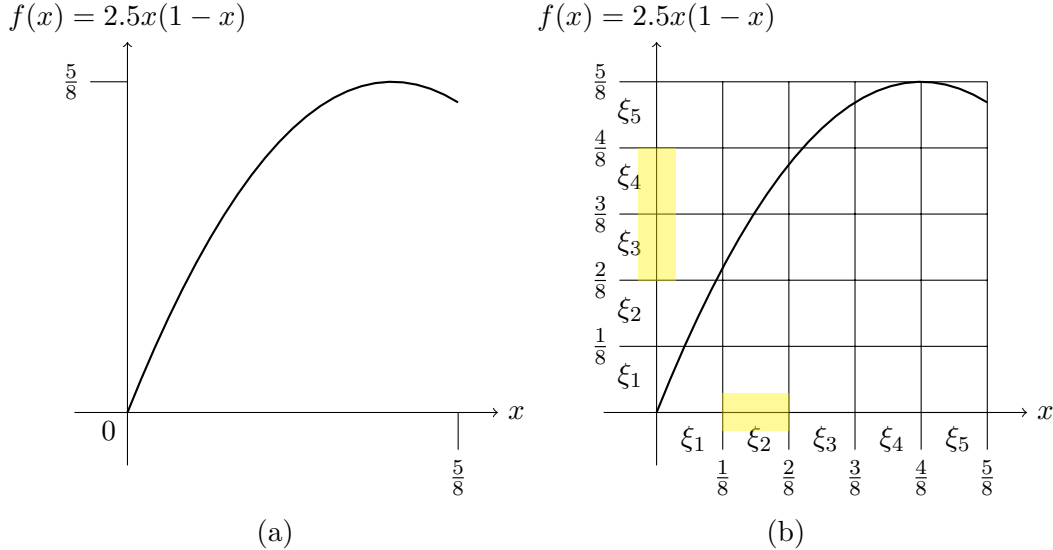

(a)   (b)

Figure 10: (a) Graph of the Logistic map on $[0, \frac{5}{8}]$; (b) Grid on the one-dimensional phase space, depicting $\mathcal{F}(\xi_2) = \{\xi_3, \xi_4\}$, or directed edges $\xi_2 \to \xi_3$ and $\xi_2 \to \xi_4$ from the vantage of graph theory

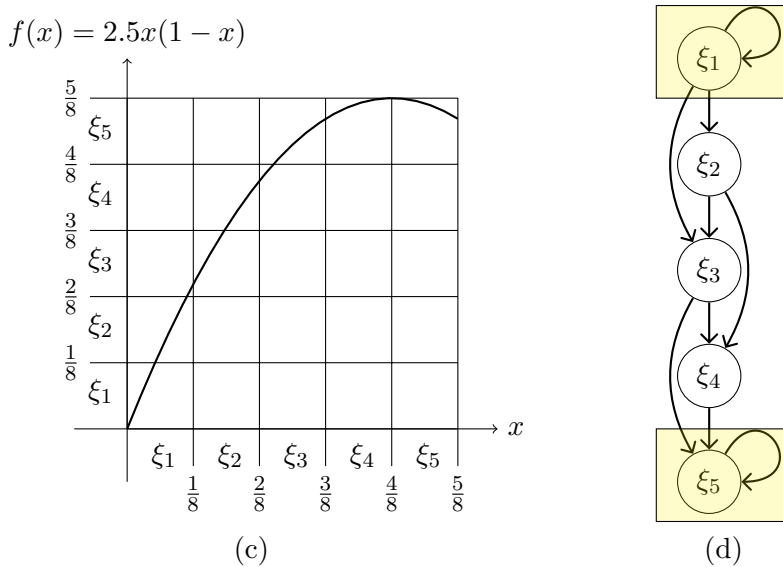(c)   (d)

Figure 11: (c) Logistic map and grid $\mathcal{X}$; (d) Directed graph representation and strongly connected path components

8

4.3 Constructing a Database

We use the term 'database' to refer to the construction of a *database of dynamical systems*. That is, one can discretize the parameter space using a grid $\mathcal{Z}$, then for each grid element $\zeta \in \mathcal{Z}$ one obtains a different combinatorial description, $\mathcal{F}_\zeta$, of the dynamics over ζ; a dynamical system $\mathcal{F}_\zeta$ for each $\zeta \in \mathcal{Z}$. Ultimately, for each $\zeta \in \mathcal{Z}$ we have an associated Morse graph, $\mathsf{MG}(\mathcal{F}_\zeta)$. Using techniques such as interval arithmetic [1, 15], one can guarantee that for each particular parameter $z \in \zeta$, $\mathcal{F}_\zeta$ is a valid dynamical representation for f_z. The idea is depicted in Process 3.

Example 4: Construction of Multivalued Map $\mathcal{F}_\zeta$

Parameter space for logistic map with grid $\mathcal{Z}$. Highlighted is grid element $\zeta_3 \in \mathcal{Z}$, where $|\zeta_3| = [2.5, 2.75]$.
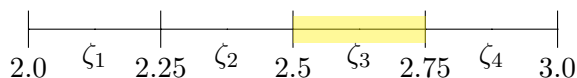
We consider an example where we discretize the parameter space and form a multivalued map for $\zeta \in \mathcal{Z}$ with $|\zeta_3| = [2.5, 2.75]$. We again use the logistic map, now on the domain [0,1]:

$$f(x) : [0, 1] \to [0, 1] = rx(1 - x)$$

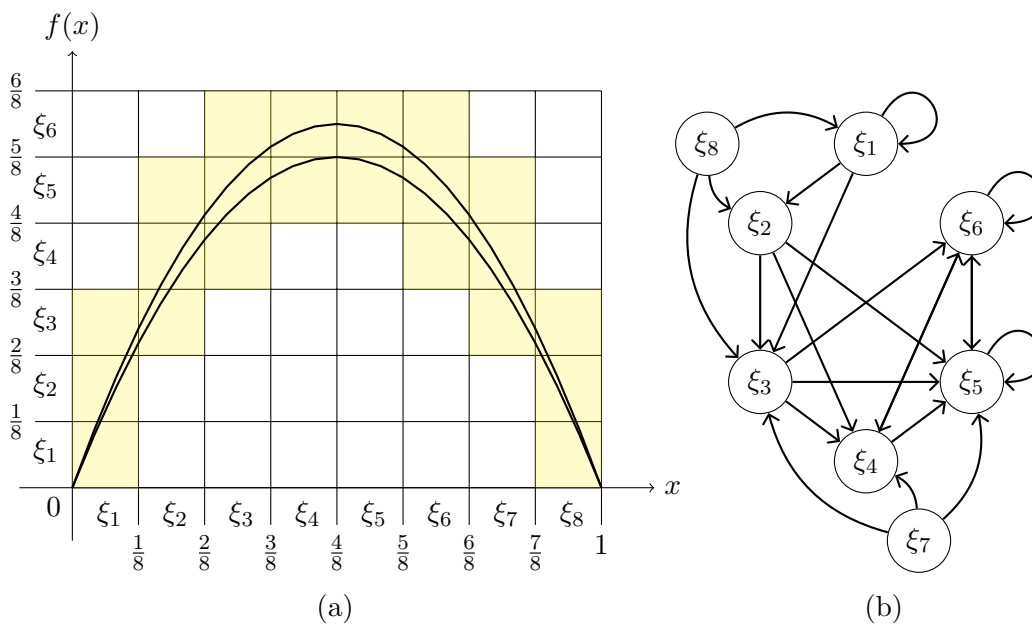(a)   (b)

Figure 12: (a) Graph of the logistic map on $[0, 1]$ for $r = 2.5$ and $r = 2.75$; the region between the parabolas corresponds to $r \in [2.5, 2.75]$. For each grid element in the domain the multivalued map of the grid element under $\mathcal{F}$ is the set of intervals in the range indicated by the highlighted boxes above the domain grid element. (b) The corresponding directed graph representation. Notice that the entire graph forms a strongly connected component.

4.3.1 Comparing Dynamics

One of the most important aspects of the Conley-Morse database is that one can infer how the dynamics at particular grid element ζ relate to the dynamics at neighboring grid elements.

9

Process 3: Constructing a Database

For each $\zeta \in \mathcal{Z}$ there is an associated Morse graph, $\mathsf{MG}(\mathcal{F}_\zeta)$. This provides a combinatorial characterization of the dynamics at each grid element for a grid over a bounded region of the parameter space.
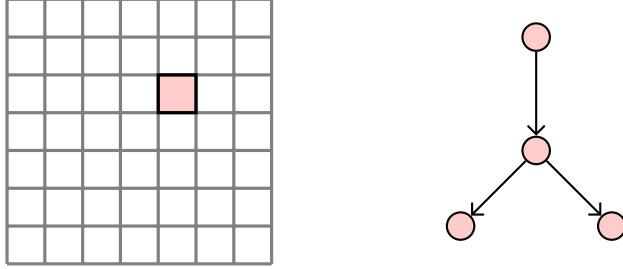

Figure 13: Consider a grid $\mathcal{Z}$ on parameter space Z with particular grid element ζ highlighted and an associated Morse graph, $\mathsf{MG}(\mathcal{F}_\zeta)$

Consider $\mathsf{MG}(\mathcal{F}_\zeta)$ and $\mathsf{MG}(\mathcal{F}_{\zeta'})$, two Morse graphs of neighboring parameter elements ζ and ζ' as shown in Figure 14. Recall that each vertex or Morse set $\mathcal{M}(p)$ in a Morse graph $\mathsf{MG}(\mathcal{F}_\zeta)$ corresponds to a strongly connected path component, which has a geometric realization in the phase space, which we denote $|\mathcal{M}(p)|$. One way to relate $\mathsf{MG}(\mathcal{F}_\zeta)$ and $\mathsf{MG}(\mathcal{F}_{\zeta'})$ is through considering the realizations of the their Morse sets in the phase space. In particular, one can define a bipartite graph with an undirected edge $\mathcal{M}_\zeta(p) \to \mathcal{M}_{\zeta'}(q)$ if $|\mathcal{M}_{\zeta'}(p)|$ and $|\mathcal{M}_\zeta(q)|$ intersect nontrivially in the phase space. Such a bipartite graph is shown in Figure 15. We call this graph the *clutching graph* and denote it $\mathcal{J}(\mathcal{F}_\zeta, \mathcal{F}_{\zeta'})$.

In particular, if the edges of the clutching graph define a directed graph isomorphism then we consider $\mathsf{MG}(\mathcal{F}_\zeta)$ and $\mathsf{MG}(\mathcal{F}_{\zeta'})$ to be in the same equivalence class. The reason we define equivalence of Morse graphs in this fashion we be clear after discussing the properties of the Conley index in a later section.

Process 4: Defining the Clutching Function

Consider $\mathsf{MG}(\mathcal{F}_\zeta)$ and $\mathsf{MG}(\mathcal{F}_{\zeta'})$, two Morse graphs of neighboring parameter elements ζ and ζ' as shown in Figure 14. The method with which we relate $\mathsf{MG}(\mathcal{F}_\zeta)$ and $\mathsf{MG}(\mathcal{F}_{\zeta'})$ is through the pairwise intersection of their Morse sets in phase space. In particular, one can define a bipartite graph with an undirected edge $\mathcal{M}_\zeta(p) \rightarrow \mathcal{M}_{\zeta'}(q)$ if $|\mathcal{M}(p)|$ and $|\mathcal{M}_\zeta(q)|$ intersect nontrivially in the phase space.
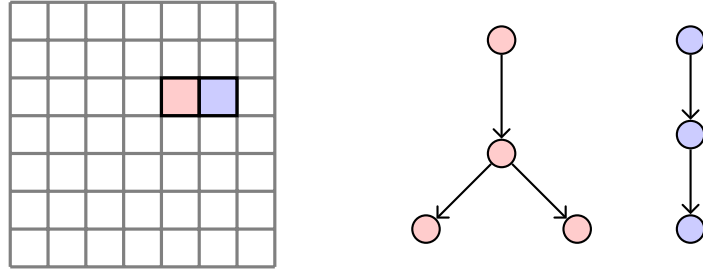

Figure 14: Consider a grid $\mathcal{Z}$ on parameter space Z; Morse graphs for ζ, ζ'

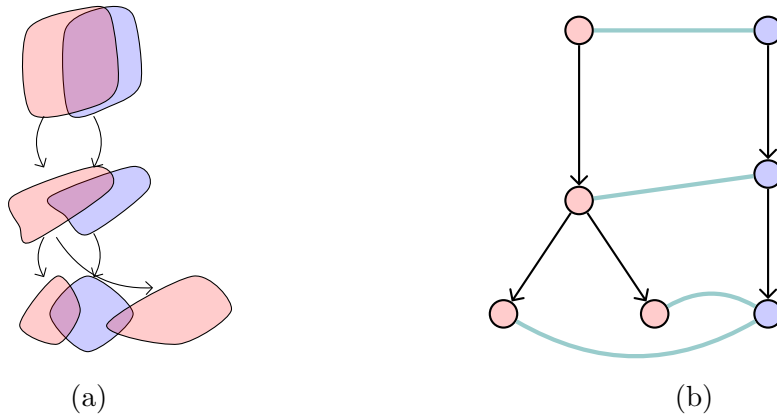(a)   (b)

Figure 15: (a) A cartoon of the realization of the Morse sets in the phase space with small arrows representing the gradient-like dynamics; (b) Intersection of the Morse sets in phase space induces a bipartite graph between the two Morse graphs

5 Discrete and Continuous Time Dynamics

In this section we introduce some ideas from classical dynamical systems theory. A dynamical system typically consists of three ingredients: a setting in which the dynamical behavior takes place, such as the real line or the circle; a mathematical rule or description which specifies the evolution of the system; an initial condition or state from which the system starts.

The basic questions of dynamical systems are qualitative: for a particular initial condition, what happens to the system in the long run? How does this long run behavior depend upon the initial condition? On the parameters within the description? On the properties of the space on which the system is defined?

In dynamics, time is either discrete (maps) or continuous (differential equations). To keep in line with our previous Examples, we will introduce the theory first for discrete time. We begin with the logistic map as presented in Example 3: a very basic example, yet one which exhibits extremely complicated dynamics.

As previously described, the logistic equation is given by:

$$f(x) = rx(1-x)$$

A traditional tool of dynamical systems theory is the *bifurcation diagram*. A bifurcation diagram shows the possible long-term values (equilibria/fixed points or periodic solutions) as a function of a control parameter in the system. The bifurcation diagram for the logistic equation is depicted in Figure 16. The bifurcation diagram is obtained non-rigorously: fixing a parameter value, say $r = 4$, one chooses an initial condition x_0 and repeatedly applying function f to x_0. Letting $x_n = f^n(x_0)$, we stop the computation if x_{n+1} becomes extremely close to x_n for a long time, indicating a fixed point. We then mark this point on the bifurcation diagram. We repeat this for many initial conditions. Let $f_2(x) = f^2(x) = f(f(x))$, we repeat this entire process. Notice if we find a fixed point for f_2, this implies finding a periodic orbit x, y, such that $f(x) = y$ and $x = f(y)$. We then repeat this entire construction for as many $n \in \mathbb{N}$ is feasible.
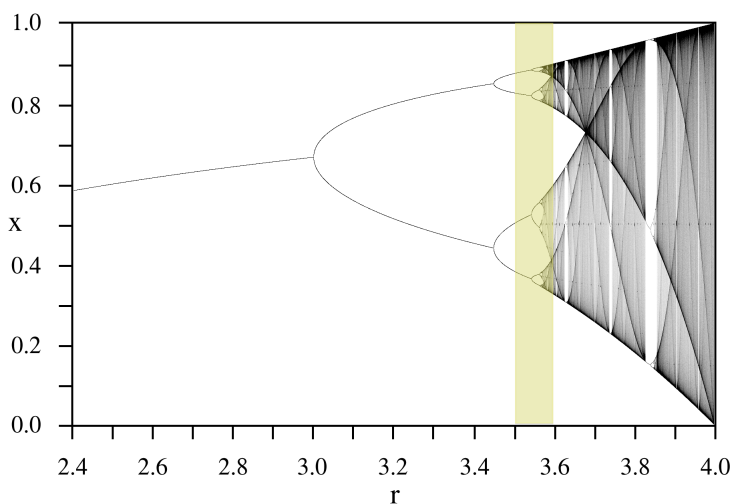

Figure 16: Even if a field biologist can measure birth rate within one decimal place, any computation can suggest the wrong dynamic.

Taking the logistic function with $r = 4$, we obtain

$$f(x) = 4x(1-x)$$

One can see that for $r = 4$ the bifurcation diagram shows very complex behavior - in fact chaotic behavior. That is, for even very similar initial condition can result in very different long-term behavior. As any real number has a finite approximation in terms of a computer, chaos has a important interplay with numerical analysis. To illustrate the affect chaos has upon numerical representation, consider the iterates $x_{n+1} = f(x_n)$. We consider two similar representations of $\frac{1}{3}$, and the iterates of f upon them.

$x_0 = 0.333333333333333$ $x_0 = 0.333333333333330$

$x_1 = 0.888888888888889$ $x_1 = 0.888888888888885$

$x_2 = 0.395061728395062$ $x_2 = 0.395061728395075$

$x_3 = 0.955951836610273$ $x_3 = 0.955951836610284$

$\vdots$ $\vdots$

$x_{15} = 0.695026128241317$ $x_{15} = 0.695026128347429$

$\vdots$ $\vdots$

$x_{49} = 0.07160322456580$ $x_{49} = 0.906436654059206$

As can be seen, after 49 iterations the difference between iterates is nearly the size of the domain. Therefore any individual orbits one computes are almost certainly not correct.

As mentioned previously, models of physical processes are often framed in terms of continuous state spaces. Much of the origins of dynamical systems theory has shown that the behavior of single trajectories can be extremely complex and may be too difficult to study independently. Instead, the traditional focus is on the invariant sets. However, the theoretical work of the last century makes clear that invariant sets can possess structure on all spatial and temporal scales and furthermore that these structures can vary dramatically over parameter sets. In particular, we highlight the following results:

1. The existence of chaos implies a sensitivity upon the initial conditions, thus the traditional focus on invariant sets instead of single trajectories

2. Structural stability is not generic - roughly the parameter sets which do not exhibit chaotic behavior are not generic - that is, it is not a rule that in general most systems are not chaotic, but rather chaos can be common (in both a topological and measure theoretic sense)

3. Both of these imply that global dynamical structures can change at all scales in parameter and phase space, leading to the fact that any computation may suggest the wrong dynamics

5.1 The Conley Theory

In its most basic form the Conley Theory consists of two components: decomposition and reconstruction. That is, one decomposes the dynamics by first isolating recurrent sets (computing the

local dynamics) and determining an admissible partial order upon them (inferring the global dynamics). Then one reconstructs the dynamics of the isolated recurrent sets using the Conley index - an algebraic topological invariant of the recurrent sets.

To keep technicalities to a minimum, we consider a multiparameter dynamical system given in the form of a continuous map,

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$
$$(x, z) \mapsto f_z(x) := f(x, z)$$

We will again let X denote a closed and bounded of $\mathbb{R}^n$ which contains the dynamics of interest. Recall an invariant set is a set $S_z \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $f(S_z) = S_z$. However, invariant sets play a subsidiary role in the Conley Theory. One of the building blocks of Conley's theory is the isolating neighborhood: compact sets $N \subset X$ such that

$$\text{Inv}(N, f_z) \subset \text{int}(N)$$

where $\text{Inv}(N, f_z)$ denotes the maximal invariant of N, i.e. the largest invariant contained in N. More precisely, $\text{Inv}(N, f_z)$ is the set of $x \in N$ such that there exists an orbit of x that is contained in N. An orbit of x is a set of points indexed by the integers $\mathbb{Z}$, $\{x_n\}_{-\infty}^{\infty}$, such that $f(x_n) = x_{n+1}$ and $x_0 = x$. An isolating neighborhood is aptly named as it isolates its associated maximal invariant set.

Figure 17 illustrates how the maximal invariant set can be isolated by the isolating neighborhood, as well as an invariant set which is not isolated.

(a)   (b)

Figure 17: (a) The maximal invariant set $\text{Inv}(N, f_z)$ is isolated by the isolating neighborhood by being contained within $\text{int}(N)$; (b) The maximal invariant set $\text{Inv}(N, f_z)$, a periodic orbit or invariant circle, which is not isolated by the isolating neighborhood.

Observe that if N is an isolating neighborhood then $\text{Inv}(N, f_z) \cap \partial N = \emptyset$, where ∂N is the topological boundary of N. The primary significance of the isolating neighborhood is that it is robust with respect to perturbations of the parameters. Here *robust* means that if N is an isolating neighborhood for f_{z_0}, it will be an isolating neighborhood for all f_z when z is sufficiently close to z_0. This is illustrated in Example 5. The other purpose of isolating neighborhoods serve to localize the dynamics being considered. Furthermore, isolating neighborhoods are indeed readily computable as will be shown later in this section.

5.2 Outer Approximation

The key idea to constructing a combinatorial multivalued map is known as an *outer approximation*. Examples of outer approximations can be seen in Process 2 and Example 5. We give a precise definition here as it is relevant to applications as it is the key assumption for many of the theorems regarding the database.

Example 5: Robustness of the Conley Index

We return to our logistic map example $f(x) = rx(1-x)$. Once again we consider the graphs for $r = 2.5$ and $r = 2.75$, which we denote $f_{2.5}$ and $f_{2.75}$.
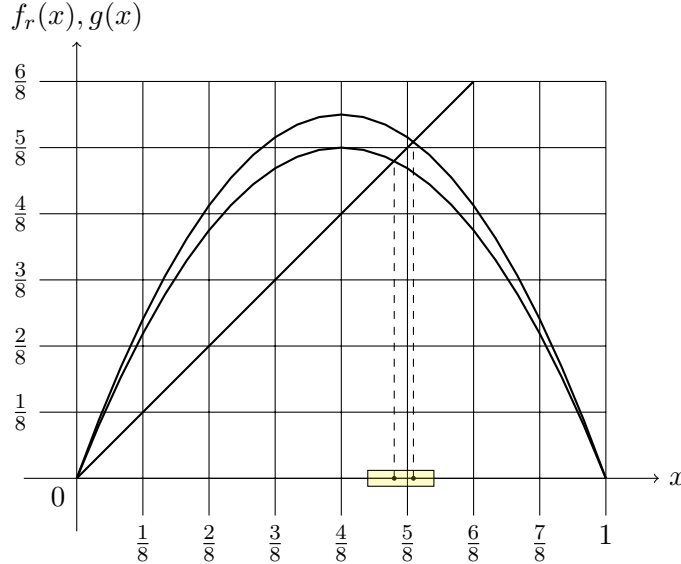

Figure 18: Graph of the logistic map on $[0,1]$ for $r = 2.5$ and $r = 2.75$ and $g(x) = x$. From the bifurcation diagram, Figure 16, we have that the only invariant sets for $f_{2.5}$ and $f_{2.7}$ are fixed points; solutions to the equation $f_r(x) = x$. These are easily spotted as the location that the graphs of $f_{2.5}$ and $f_{2.75}$ intersect the diagonal, i.e. the graph of $g(x)$. Here the highlighted region depicts an isolating neighborhood $N = [.55, .675]$ for $f_{2.5}$. $\text{Inv}(N, f_{2.5})$ is the fixed point at $x = .6$. Notice that N also serves as an isolating neighborhood for f_r where r is close to 2.5; in particular N is an isolating neighborhood for the map $f_{2.75}$, as the fixed point of $f_{2.75}$ does not intersect the boundary of N.

Definition 5.1. Consider continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ and compact subset $X \subset \mathbb{R}^n$. Let $\mathcal{X}$ be a grid on X. A combinatorial multivalued map $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ is an *outer approximation* of f, if

$$f(\xi) \subset \text{int}(|\mathcal{F}(\xi)|) \text{ for all } \xi \in \mathcal{X}$$

The main point is that the image of of a grid element under f is contained in the interior of the image under the multivalued map. The following Proposition is key in that it shows that isolating neighborhoods are readily computable via outer approximations:

Proposition 5.2. Let $\{\mathcal{M}_\zeta(p) \subset \mathcal{X} \mid p \in \mathsf{P}_\zeta\}$ be the combinatorial Morse decomposition for the outer approximation $\mathcal{F}_\zeta : \mathcal{X} \rightrightarrows \mathcal{X}$ of F_ζ. Then for all $p \in \mathsf{P}_\zeta$, $|\mathcal{M}_\zeta(p)|$ is an isolating neighborhood for f_z for all $z \in \zeta$.

Proposition 5.2 says that to compute an isolating neighborhood we only need to compute $\mathcal{M}_\zeta(p)$, the corresponding SCPC in the graph $\mathcal{F}_\zeta$, and consider its realization in the phase space $\mathbb{R}^n$.

5.3 The Conley Index

The Conley index is an algebraic topological invariant of isolated invariant sets. That is, it is algebraic topological in the sense that it is an equivalence class of an induced map on an abstract

space and invariant in the sense that isolated invariant sets isolated which can be isolated with the same isolating neighborhood have equivalent Conley indices. One can compute an isolating neighborhood without knowing the isolated invariant set (for instance, with the notion of an outer approximation). Thus, in general one has access to an isolating neighborhood and no prior knowledge of the isolated invariant set. It is in this way that the Conley index helps to understand or 'reconstruct' the dynamics of the isolated invariant set. The process of defining the Conley index is quite involved, and we will give a casual overview. The reader may consult [4, 7] for more details.

Before describing the index, there are three important aspects of the Conley index we will highlight [4]:

N1 One can associate a Conley index to any isolating neighborhood.

N2 If N and N' are isolating neighborhoods and $\mathrm{Inv}(N, f_z) = \mathrm{Inv}(N', f_z)$, then they have the same Conley index.

N3 If N is an isolating neighborhood for all z in a (path connected) subset of the parameter space Z, then the Conley index associated with N is the same for all f_z. That is, the index is robust.

The Conley index is constructed in the following manner: take an isolating neighborhood and the 'exit set' or the set of points that are leaving the neighborhood. We denote the exit set by L. The pair (N, L) is called an *index pair*.[4]

The first ingredient of the Conley index is the space obtained by collapsing the exit set to a single point. The resulting space is a 'quotient space' denoted N/L, where the collapsed exit set is now a distinguished point denoted by $[L]$. An example of forming the quotient space N/L is discussed in Procedure 5. The map f then induces a continuous map $f_{N/L} : N/L \to N/L$. Unfortunately, this map itself is not an invariant, but taking a particular equivalence class (shift equivalence [4, 7]) of the map is an invariant.

The most basic and fundamental result is the following:

If the Conley index of N is not trivial, then $\mathrm{Inv}(N, f) \neq \emptyset$.

Here to be trivial we mean not equivalent to the zero map under shift equivalence. This shows that if we compute the index or show it is not trivial we may conclude that the isolated invariant set is not empty.

The following Proposition shows that it is straightforward to compute index pairs.

Proposition 5.3. *Let* $\{\mathcal{M}_\zeta(p) \subset \mathcal{X} \mid p \in \mathsf{P}_\zeta\}$ *be the combinatorial Morse decomposition for the outer approximation* $\mathcal{F}_\zeta : \mathcal{X} \rightrightarrows \mathcal{X}$ *of* F_ζ. *Assume* $|S_\zeta| \subset \mathrm{int}(X)$. *Let the pair* (N, L) *be defined by*

$$N := \left|\mathcal{F}_\zeta(\mathcal{M}_\zeta(p))\right| \quad and \quad L := \left|\mathcal{F}_\zeta(\mathcal{M}_\zeta(p)) \setminus \mathcal{M}_\zeta(p)\right|$$

Then for all $z \in \zeta$, (N, L) *is an index pair for* f_z.

Proposition 5.3 shows that to compute an index pair of a Morse set $\mathcal{M}_\zeta(p)$ it suffices to compute its forward image under the multivalued map $\mathcal{F}$, or the set of vertices in the graph $\mathcal{F}$ which have an in edge from some vertex in $\mathcal{M}_\zeta$.

---

[4]There are more technical restrictions on index pairs which we do not mention here. See [7] or Appendix B for details.

1. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{1}{2}y \end{pmatrix}$$

2. The fixed point of f at $x = (0,0)$ exhibited in Figure 19 is the isolated invariant set of the depicted isolating neighborhood.
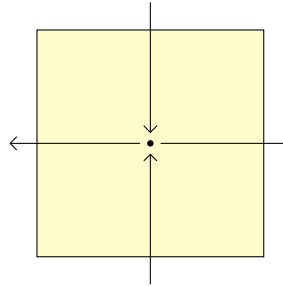

Figure 19: The maximal invariant set $\mathrm{Inv}(N, f_z) = (0,0)$ is isolated by the isolating neighborhood by being contained within $\mathrm{int}(N)$

3. Construct the exit set of the isolating neighborhood: the set of points which leave the neighborhood after application of f; this is pictured in Figure 20.
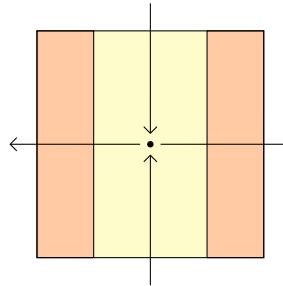

Figure 20: The exit set of the isolating neighborhood

4. The quotient space N/L obtained by collapsing the exit set encodes some structure of the invariant set. For instance, in this case the space is topologically equivalent to a circle.
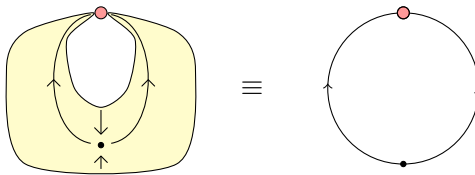

Figure 21: Collapsing the exit set to a point

5.4 Conley-Morse Graphs

We have previously defined the objects Morse graphs. Armed with the Conley index, we have a tool for characterizing the local dynamics in each Morse set, or vertex in a Morse graph. Accordingly we define a new object: the *Conley-Morse Graph*, denoted CMG_ζ of $\mathcal{F}_\zeta$ consists of the Morse graph MG_ζ of $\mathcal{F}_\zeta$ along with the Conley index associated with each Morse set $\mathcal{M}_\zeta(p)$, $p \in \mathsf{P}_\zeta$.

To construct the database we can consider all the concepts developed in the last few sections, within Conley-Morse graphs replacing Morse graphs. The following Proposition clarifies our definition of equivalence in the last section [4].

Proposition 5.4. *Assume there is a unique edge (p, q) in the clutching graph $\mathcal{J}(\zeta, \zeta')$ that has either p or q as its endpoint. That is, p and q have no other edges. Then the Conley index of $|M_\zeta(p)|$ under F_ζ is equivalent to the Conley index of $|M_\zeta(p)|$ under $F_{\zeta'}$.*

As a Corollary to the above Proposition, we have that if two Morse graphs are equivalent (using the definition we developed in the last section), then the Conley indices of each of the Morse sets are equivalent under the bijection.

To summarize the construction of the last few sections, to each Conley-Morse graph equivalence class, we associate three distinct types of information:

- The Morse graph, which provides information about the global dynamics, that is, the non-recurrent dynamics. The partial order obtained from the Morse graph constraints the non-recurrent dynamics. Morse graphs can be understood as giving a schematic picture of the dynamics in phase space away from any recurrent dynamics.

- The Conely indices of the Morse sets, which provide information about the structure of the recurrent (local) dynamics.

- The set of parameter grid elements whose Conley-Morse graph belong to the Conley-Morse equivalence class. One can understood these equivalence classes as identifying the region in parameter space where the identified recurrent and non-recurrent dynamics occurs at the scale of the computation.

The *continuation graph* is the graph whose nodes consist of the Conley-Morse graph equivalence classes and whose edges consist of the clutching graph information between representative Conley-Morse graphs. This is shown in Section 6.

6 Application: Leslie Model

We will provide a overview of the database applied to a population model.[5]

Mathematically, the two-dimensional Leslie model is defined to be a map $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x_1 + \theta_2 x_2)e^{\alpha(x_1 + x_d)} \\ p x_1 \end{bmatrix}$$

The Leslie model, as well as a discussion regarding its importance in population biology, are given in detail in [16]. In the model, the population is partitioned into two generations, each with population x_i and an associated reproduction rate, θ_i. The nonlinearity stems from the assumption that fertility decreases exponentially with the total size of the population. For this example we will fix $p = 0.7$ and $\alpha = -0.1$.

---

[5]A more detailed example on how to use the software can be found at `http://chomp.rutgers.edu/Projects/Databases_for_the_Global_Dynamics/software/CMD_example.html`.

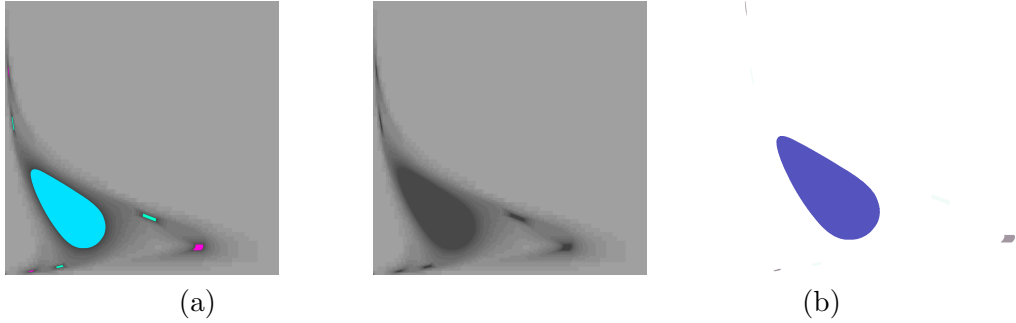(a)   (b)

Figure 22: (a) Morse sets in the phase space of the Leslie Model, (b) Adaptive grid decomposition

Figure 23 is taken from [4] and shows the Conley-Morse database visualization software package available at `chomp.rutgers.edu`, analyzing the Leslie population model [4, 16].[6]
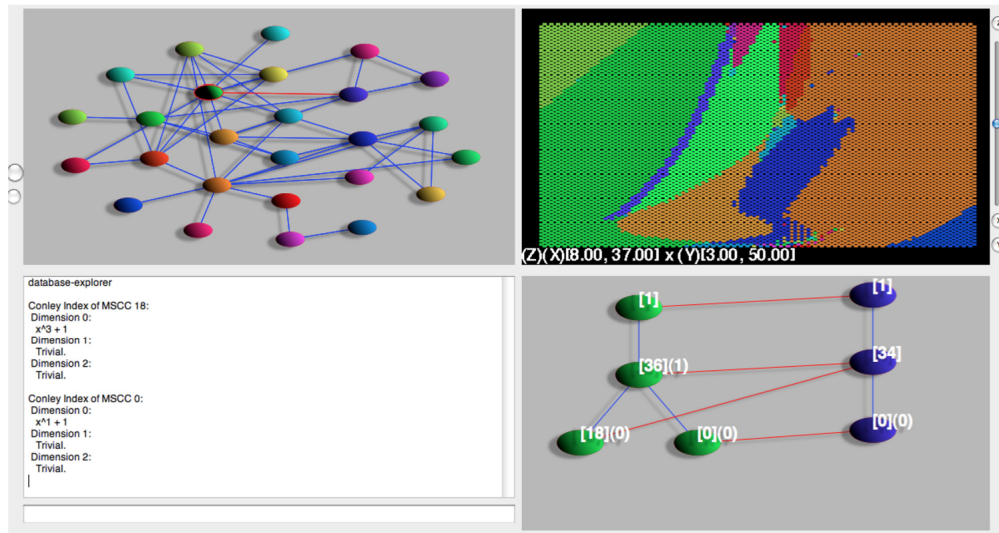

Figure 23: Database information concerning dynamics for the overcompensatory Leslie model [16]. (Upper left) Continuation Graph: Each node corresponds to a Conley-Morse graph equivalence class. Each edge corresponds to a clutching graph between Conley-Morse graphs. (Upper right) Parameter space divided into regions corresponding to Conley-Morse graph equivalence classes. Color coding of parameter space matches the color coding of the nodes in the continuation graph. (Lower right) Clutching graph between two Conley-Morse graphs. This clutching graph corresponds to the highlighted (red) edge in the continuation graph.

---

[6]A much more detailed explanation of the database explorer can be found at `http://chomp.rutgers.edu/Projects/Databases_for_the_Global_Dynamics/software/CMD_database_explorer.html`

Appendix A Notation

Symbol **Meaning**

$\in$ $x \in N$ denotes x is belongs to the set N, or x is an 'element of' N

$\subset$ $S \subset X$ denotes that S is a subset of X; if $x \in S$ then $x \in X$

∂ ∂N denotes the boundary of N

$\text{cl}(\cdot)$ $\text{cl}(N)$ denotes the closure of the N

$\text{int}(\cdot)$ $\text{int}(N)$ denotes the interior of the N

$\setminus$ $N \setminus L$ is the set of points in N that are not in L

$:=$ $N := X$ defines the symbol N to be the object X

$|\cdot|$ If $\mathcal{M} \subset \mathcal{X}$, $\mathcal{X}$ a grid on X, $|\mathcal{M}|$ is the realization in X - that is, $|\mathcal{M}|$ is a subset of X

Appendix B Definitions

Definition B.1 (Boundary). For a set $N \subset X$, the boundary of N, denoted ∂N, is the set of points p of X such that every neighborhood of p contains at least one point of N and at least one point not of N.

Definition B.2 (Closure). For a set $N \subset X$, the closure of N, denoted $\text{cl}(N)$, is N union its boundary, i.e. the set $N \cup \partial N$.

Definition B.3 (Interior). For a set $N \subset X$, the interior of N, denoted $\text{int}(N)$, is the set of points in N that do not belong to ∂N, the boundary of N.

The point y is on the boundary of N since any neighborhood of y intersects N and the complement of N. The point x belongs to the interior of N.

Definition B.4 (Index Pair). Let S be an isolated invariant set and suppose (N, L) with $L \subset N$ are a pair of compact sets contained in the interior of the domain of f. The pair (N, L) is called an *index pair* for S provided N and L are each the closures of their interiors and
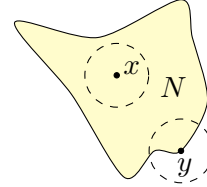
1. $cl(N \setminus L)$ is an isolating neighborhood of S,

2. L is a neighborhood of N^- in N,

3. $f(L) \cap \text{cl}(N \setminus L) = \emptyset$.

(where above $N \setminus L$ is the set of points in N not in L)

Definition B.5 (Maximal Invariant Set of a Graph). The maximal invariant set, S, of a graph $G = (V, E)$ is the maximal set where $S \subset V$ and $\{w \in V \mid$ there exists an edge $v \to w$ for $v \in S\} = S$.

Definition B.6 (Partial Order). A *partial order* is a binary relation $\leq$ over a set P such for $p, q, r \in \mathsf{P}$, we have:

I. $p \leq p$ for all $a \in \mathsf{P}$ (reflexivity)

II. if $p \leq q$ and $q \leq p$ then $p = q$ (antisymmetry)

III. if $p \leq q$ and $q \leq r$ then $p \leq r$ (transitivity)

Appendix C   Shift Equivalence

Shift equivalence is most easily defined in a general setting. Consider two functions a, b where

$$a : V \to V \quad \text{and} \quad b : W \to W$$

where V and W are objects in some category $\mathcal{K}$ and a and b are endomorphisms. For instance, V and W are both either finitely generated abelian groups or finite dimensional vector spaces, and a and b are correspondingly group endomorphisms or linear maps.

Definition C.1 (Shift Equivalence). *The maps a and b are* shift equivalent *if there exist morphisms*

$$r : V \to W \quad and \quad s : W \to V$$

such that

$$b \circ r = r \circ a \quad and \quad s \circ b = a \circ s$$

and a positive integer n such that

$$s \circ r = a^n \quad and \quad r \circ s = b^n$$

Shift equivalence arises frequently in dynamical systems as a natural equivalence relation. For instance, note that if a and b are homeomorphisms on topological spaces then they are shift equivalent if and only if they are topologically conjugate [7].

References

[1] Z. Arai, W. Kalies, H. Kokubu, K. Mischaikow, H. Oka, and P. Pilarczyk, *A Database Schema for the Analysis of Global Dynamics of Multiparameter Systems*, SIAM Journal of Applied Dynamical Systems 8, 2009.

[2] D. Aubin, and A. Dahan Dalmedico, *Writing the History of Dynamical Systems and Chaos: Longue Dure and Revolution, Disciplines and Cultures, Historia Mathematica*, Volume 29, Issue 3, 2002.

[3] H. Ban, and W. Kalies, *A Computational Approach to Conley's Decomposition Theorem*, J. Comp. Nonlinear Dynamics, 2006.

[4] J. Bush, M. Gameiro, S. Harker, H. Kokubu, K. Mischaikow, I. Obayashi, and P. Pilarczyk, *Combinatorial-topological framework for the analysis of global dynamics*, Chaos, 22, 2012.

[5] C. Conley, *Isolated Invariant sets and the Morse Index*, in CMBS Regional Conference Series in Mathematics, American Mathematical Society, 1978, Vol. 38.

[6] T. Cormen, C. Leiserson, and R. Rivest, *Introduction to Algorithms*, The MIT Electrical Engineering and Computer Science Series, MIT, Cambridge, MA, 1990.

[7] J. Franks and D. Richeson, *Shift equivalence and the Conley Index*, Transactions of the American Mathematical Society, 352, 2000.

[8] J. Gleick, *Chaos: Making a New Science*, Penguin Books, New York, NY, USA, 1987.

[9] K. Mischaikow, *The Conley Index Theory: Some Recent Developments*, In Dynamical Systems, Springer, 1995.

[10] K. Mischaikow and M. Mrozek, *Conley Index*, in Handbook of Dynamical Systems, Volume 2, North-Holland, Amsterdam, 2002.

[11] J. Munkres, *Topology*, Prentice Hall, Incorporated, 2000.

[12] J. Palis, and F. Takens, *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations: Fractal Dimensions and Infinitely Many Attractors in Dynamics*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.

[13] D. Salamon, *Morse Theory, the Conley Index and Floer Homology*, Bulletin of the London Mathematical Society, 1990.

[14] S. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering*. Perseus Books Group, 1994.

[15] W. Tucker, *Validated Numerics: A Short Introduction to Rigorous Computations*, Princeton University Press, 2011.

[16] I. Ugarcovici and H. Weiss, *Chaotic Dynamics of a Nonlinear Density Dependent Population Model*, Nonlinearity 17, 2004.