A COMPLEMENTATION THEOREM FOR PERFECT MATCHINGS OF GRAPHS HAVING A CELLULAR COMPLETION

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Abstract. A cellular graph is a graph whose edges can be partitioned into 4-cycles (called cells) so that each vertex is contained in at most two cells. We present a "Complementation Theorem" for the number of matchings of certain subgraphs of cellular graphs. This generalizes the main result of [2]. As applications of the Complementation Theorem we obtain a new proof of Stanley's multivariate version of the Aztec diamond theorem, a weighted generalization of a result of Knuth [7] concerning spanning trees of Aztec diamond graphs, a combinatorial proof of Yang's enumeration [17] of matchings of fortress graphs and direct proofs for certain identities of Jockusch and Propp [6].

1. Introduction

A *perfect matching* of a graph is a collection of vertex-disjoint edges that are collectively incident to all vertices. We will often refer to a perfect matching simply as a *matching*. Denote by $M(G)$ the number of matchings of the graph G. In case G is weighted (i.e., is equipped with a weight function on the edges), the weight of a matching μ is the product of the weights of the edges contained in μ; in this case $M(G)$ is defined to be the sum of the weights of all matchings of G, and is called the *matching generating function* of G.

The *Aztec diamond of order n*, denoted AD_n, is defined to be the graph whose vertices are the white squares of a $(2n+1) \times (2n+1)$ chessboard with black corners, and whose edges connect precisely those pairs of white squares that are diagonally adjacent (Figure 3.1 illustrates AD_3).

In [**5**] there are presented four proofs of the fact that $M(AD_n) = 2^{n(n+1)/2}$. The recurrence $M(AD_n) = 2^n M(AD_{n-1})$ (which clearly implies the latter formula) has been generalized in [**2**] to bipartite cellular graphs (a graph is *cellular* if its edges can be partitioned into 4-cycles so that each vertex is contained in at most two 4-cycles of the partition). Namely, if G is a bipartite cellular graph, then it is proved in [**2**] that $M(G) = 2^{\nu(G)} M(G')$, where $\nu(G)$ is a certain statistic and G' is an easily constructible subgraph of G.

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In this paper we prove a generalization (which we call the Complementation Theorem) of this result to weighted graphs admitting a (not necessarily bipartite) cellular completion (see Section 2 for the definition). This allows us to obtain a number of new applications.

First, we obtain a new proof for Stanley's multivariate Aztec diamond theorem [**12**]. We also give a new multivariate version of the Aztec diamond theorem, one having $4n+1$ free parameters for the diamond of order n. The edge weighting we employ is quite different from the ones used in the two previous multivariate versions (due to Stanley and Yang, respectively), both of which involve $4n$ free parameters in the order n case.

Second, as an immediate corollary of the Complementation Theorem we obtain direct combinatorial proofs for some identities of Jockusch and Propp [**6**] which relate the numbers of matchings of three kinds of quartered Aztec diamonds. In at least one case, this appears to be the first combinatorial proof.

Third, employing also a construction of Temperley that relates spanning trees to perfect matchings, we give a combinatorial proof of Stanley's ex-conjecture on the number of spanning trees of the even and odd Aztec diamonds, first proved by Knuth [**7**] using linear algebra. Our argument yields in fact a weighted generalization of Knuth's result, thus proving a particular case of a more general conjecture of Chow (recently, Chow has proved his general conjecture [**1**]).

Fourth, we use the Complementation Theorem to refine a formula due to Mills, Robbins and Rumsey [**11**] for a certain weighted count of alternating sign matrices.

And fifth, we present a new way of enumerating the perfect matchings of the fortress graphs considered by Yang in [**17**]. This appears to be the first combinatorial proof of Yang's formulas.

2. The Complementation Theorem

Let G be a graph with vertex set $V(G)$. A *cellular graph* is a finite graph whose edges can be partitioned into 4-cycles — the *cells* of the graph — such that each vertex is contained in at most two cells. A *weighted cellular graph* is a cellular graph equipped with a weight function on the edges.

Let c_0 be a cell and consider two opposite vertices x_0 and y_0 of c_0. By definition, x_0 is contained in at most one other cell besides c_0. If such a cell c_1 exists, let x_1 be its vertex opposite x_0. Then x_1 in turn is contained in at most one cell other than c_1; if such a cell exists denote by x_2 its vertex opposite x_1, and continue in this fashion. Repeat the procedure starting with y_0.

The set of cells that arise this way is said to be a *line* of G. If the sequence $\{x_i\}$ (and hence the analogous sequence $\{y_i\}$) defined above is finite, the line is called a *path*, and the last entry of each of the two sequences is called an *extremal vertex* of G. In the remaining case $\{x_i\}$ must be periodic and the line is referred to as a *cycle*.

Let $X(G)$ be the set of extremal vertices of G.

Given a graph H, a cellular graph G is said to be a *cellular completion* of H if

(i) H is an induced subgraph
(ii) $V(G) \setminus V(H) \subseteq X(G)$.

Let G be a cellular completion of the graph H. The *complement of H* (with respect to G) is defined to be the induced subgraph H' of G whose vertex set is determined by the

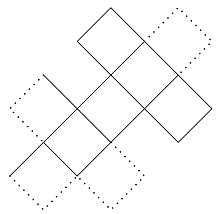


FIGURE 2.1(a)

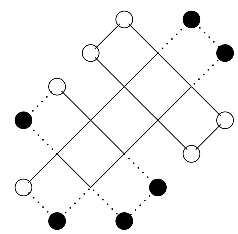


FIGURE 2.1(b)

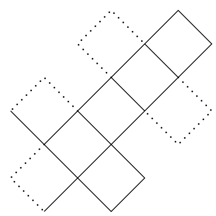


FIGURE 2.1(c)

equation $V(H')\triangle V(H) = X(G)$, where the triangle denotes symmetric difference of sets (an example is illustrated in Figure 2.1; the edges of G not contained in H (resp., H') are represented by dotted lines). In other words, $V(H')$ is the set obtained from $V(H)$ after performing the following operation at each end of every path of G: if the corresponding extremal vertex belongs to $V(H)$, remove it; otherwise, include it. We note that a graph may have more than one cellular completion (see the paragraph following the proof of Lemma 4.2). This fact will prove to be crucial in many of our results.

If G (and hence, by restriction, H) is weighted by the weight function wt, define a new weight wt' (the *complementary weight* of wt) by setting $\mathrm{wt}'(e) := \mathrm{wt}(e')$, where e' is the edge opposite e in the cell containing e. The weight on H' is defined to be the restriction of wt' to H'.

If an extremal vertex of a path L of G belongs to $V(H)$ then the path is said to be *closed* at that end; otherwise, we say it is *open* at that end. Define the *type* $\tau(L)$ of the path L to be 1 less than the number of closed ends of L; define the type of each cycle to be 0.

Let Δ be the function defined on the set of cells as follows: if the cell c has edges weighted by x, y, z and w (in cyclic order), set $\Delta(c) := xz + yw$.

THEOREM 2.1 (COMPLEMENTATION THEOREM). *Let G be a weighted cellular graph and suppose its cells can be partitioned into disjoint lines $L_1, L_2, \ldots, L_k$ so that $\Delta(c) = \Delta_i$ for all cells c along L_i, $i = 1, 2, \ldots, k$. If G is a cellular completion of the subgraph H, we have*

$$M(H) = \Delta_1^{\tau(L_1)}\Delta_2^{\tau(L_2)}\cdots\Delta_k^{\tau(L_k)}M(H').$$

As an illustration, suppose H is the graph shown in Figure 2.1(a), and consider the partition of its cells into horizontal lines. Since the value of τ on each such line is zero, we obtain that H has the same number of matchings as the graph shown in Figure 2.1(c).

Before giving the proof we need some preliminary results. Given a graph H and a cellular completion G, an *alternating sign pattern (ASP) of shape* (H, G) is an assignment of integers to the cells of G such that

(i) all entries are 1, 0 or -1

(ii) the non-zero elements along each line alternate in sign

(iii) in every path, the closest non-zero element to a closed end is a 1, and the closest non-zero element to an open end is a -1.

Denote by $ASP(H, G)$ the set of alternating sign patterns of shape (H, G).

Let G and H be as in the statement of the Complementation Theorem, and consider a matching μ of H. Write one of the numbers 1, 0 or -1 in each cell of G corresponding to the cases when the cell contains 2, 1 or 0 edges of μ. According to these numbers, we call them 1-, 0- or (-1)-cells. Let A denote the obtained pattern.

LEMMA 2.2. $A \in ASP(H, G)$.

Proof. Consider a portion of a line L of G consisting of a 1-cell c bordered on both sides by (possibly empty) sets of 0-cells. Suppose c' is a 0-cell next to c and let x be their common vertex. Since x is matched internally in c, the unique edge of μ contained in c' must lie in the "hook" of c' that points away from c, i.e., in the union of the two edges of c' not touching c. Therefore, the vertex x' of c' opposite x is matched inside c'. Hence, the next 0-cell c'' must contribute to μ with an edge contained in the hook of c'' pointing away from c. This argument can be repeated throughout both runs of 0-cells bordering c.

For any two runs of 0-cells separated by a single 1-cell we obtain this way a set of hooks on the 0-cells pointing away from the 1-cell with the property that the unique edge contributed by each 0-cell in μ is contained in the corresponding hook.

A similar argument shows that if two runs of 0-cells are separated by a single (-1)-cell, there is an induced collection of hooks on the 0-cells pointing *towards* the (-1)-cell such that the edge of μ contained in each 0-cell is contained in the hook placed on that cell.

Finally, consider the case when there is a run of 0's next to an extremal vertex of a path. If the extremal vertex is not in $V(H)$, then essentially the same argument we used in the first paragraph of this proof shows that the unique edge contributed by each 0-cell to the matching must lie in the hook pointing away from the extremal vertex. Similarly, if the extremal vertex belongs to $V(H)$, we obtain a collection of hooks pointing towards the extremal vertex and containing the restriction of μ to these 0-cells.

Regard each line of A as being built up from pieces consisting of either two maximal runs of 0's separated by a single non-zero entry or one maximal run of zeroes at an end of a path of A (the only lines of A not covered by this description are the cycles consisting entirely of zeroes; however, properties (i)–(iii) are obviously met in this case).

By the above construction, each maximal run of 0-cells is given two sets of hooks, both containing the edges contributed by the 0-cells to μ. Therefore the hooks must have nonempty intersection, and this happens only if the two sets of hooks coincide along every maximal run of 0's. This in turn is equivalent precisely to conditions (ii) and (iii) from the definition of an alternating sign pattern. □

Let $A \in ASP(H, G)$ and think of the entries of A as being written in the corresponding cells of G. For each cycle C of G consisting entirely of zeroes there are exactly two collections of disjoint hooks on the cells of C pointing along the cycle.

LEMMA 2.3. *Choose independently one of the two sets of disjoint hooks pointing along each cycle of G consisting entirely of zeroes. Then there are precisely $2^{N_+(A)}$ matchings of H compatible with the selected sets of hooks and having associated ASP A, where $N_+(A)$ denotes the number of entries of A equal to 1.*

Proof. Consider along each line of G the hooks constructed in the proof of Lemma 2.2. Note that since these hooks are determined by the non-zero entries of A and by the extremal vertices of G, this construction provides hooks on the 0-cells of each line, except for the cycles consisting entirely of zeroes. However, for each of these cycles we are given a set of disjoint hooks pointing along the cycle.

Any matching μ of H with associated ASP A and compatible with the selected hooks on the all-zero cycles must be compatible with all hooks under consideration. Let c be a 0-cell. Since there are two lines containing each cell, we have two hooks placed on c. However, these two hooks always intersect in a single edge of c. This shows that there is a unique way of specifying μ on the 0-cells so as to meet our requirements. Since there are $2^{N_+(A)}$ possibilities for the restriction of μ to the union of the 1- and (-1)-cells, we obtain the statement of the Lemma. $\square$

Proof of Theorem 2.1. Consider an alternating sign pattern of shape (H, G) and let $\mathcal{C} = \{C_1, C_2, \dots, C_l\}$ be the set of cycles of G consisting entirely of zeroes. On each cycle C_i declare one of the two sets of disjoint hooks pointing along the cycle as being distinguished (we say that this set of hooks specifies an orientation of C_i). We will denote by C_i^+ and C_i^-, respectively, the fact that the the distinguished or the non-distinguished set of hooks has been chosen on C_i. Denote by $M_A(H; C_1^{\pm}, \dots, C_l^{\pm})$ the generating function of matchings of H compatible with a specified orientation of the C_i's and having associated ASP A. We claim that

$$M_A(H; C_1^+, \dots, C_l^+) = \Delta_1^{\tau(L_1)} \Delta_2^{\tau(L_2)} \cdots \Delta_k^{\tau(L_k)} M_{-A}(H'; C_1^-, \dots, C_l^-), \tag{2.1}$$

where $-A$ denotes the pattern obtained by changing the sign of all entries in A.

To show this, note first that the map $A \mapsto -A$ is a bijection between $ASP(H, G)$ and $ASP(H', G)$. Indeed, as a consequence of the definition of the complement, G is a cellular completion of H', so $ASP(H', G)$ is well-defined. Furthermore, conditions (i) and (ii) from the definition of an alternating sign pattern are clearly invariant under interchanging 1's and (-1)'s. Moreover, since an extremal vertex of a path is contained in $V(H)$ if and only if it is not contained in $V(H')$, it follows that the effect of this interchanging on (iii) is that of replacing H by H'.

As was shown in the proof of Lemma 2.3, all $2^{N_+(A)}$ matchings in $\mathcal{M}_A(H; C_1^+, \dots, C_l^+)$ (i.e., matchings with associated ASP A compatible with the distinguished sets of hooks on the C_i's) are identical when restricted to the 0-cells. Therefore, up to a multiplicative factor equal to the product of the weights of the specified edges on the 0-cells, $M_A(H; C_1^+, \dots, C_l^+)$ is given by considering only the contribution of the 1-cells. As the lines $L_1, L_2, \dots, L_k$ partition the set of cells and since $\Delta(c) = \Delta_i$ for all cells c along L_i $(i = 1, 2, \dots, k)$, we conclude that the total contribution of the 1-cells in the matchings of $\mathcal{M}_A(H; C_1^+, \dots, C_l^+)$ is

$$\Delta_1^{N_+(L_1)}\Delta_2^{N_+(L_2)}\cdots\Delta_k^{N_+(L_k)}, \tag{2.2}$$

where $N_+(L_i)$ denotes the number of 1's in the cells of L_i.

Similarly, Lemma 2.3 implies that all matchings in $\mathcal{M}_{-A}(H'; C_1^-, \ldots, C_l^-)$ have identical restrictions to the 0-cells. Moreover, note that the specified edge e' on each 0-cell c is now the edge opposing the edge e of c participating in the matchings in $\mathcal{M}_A(H; C_1^+, \ldots, C_l^+)$. However, recall that the weight wt' on H' was defined precisely by $\mathrm{wt}'(e') = \mathrm{wt}(e)$. Therefore, the 0-cells contribute to $M_{-A}(H'; C_1^-, \ldots, C_l^-)$ by the same multiplicative factor as the one considered in the previous paragraph. On the other hand, we obtain as before that the contribution of the 1-cells is

$$\Delta_1^{N_+(-L_1)}\Delta_2^{N_+(-L_2)}\cdots\Delta_k^{N_+(-L_k)}, \tag{2.3}$$

where $-L$ denotes the restriction of $-A$ to the line L. However, $N_+(-L_i) = N_-(L_i)$ for all i. Also, $A \in ASP(H, G)$ implies that the quantity $N_+(L) - N_-(L)$ is equal to $\tau(L)$. Relation (2.1) follows then from (2.2) and (2.3).

Relation (2.1) is valid for all 2^l orientations of the cycles in $\mathcal{C}$. Therefore, since the operation of reversing the orientation of all cycles in $\mathcal{C}$ permutes the 2^l possible orientations, we deduce from relation (2.1) that

$$M_A(H) = \Delta_1^{\tau(L_1)} \cdots \Delta_k^{\tau(L_k)} M_{-A}(H'),$$

where $M_A(H)$ denotes the generating function of matchings of H with corresponding ASP A. Summing over $A \in ASP(H, G)$ and using the fact that $A \mapsto -A$ is a bijection between $ASP(H, G)$ and $ASP(H', G)$, we obtain by Lemma 2.2 the statement of the theorem. □

Remark 2.4. It is interesting to ask under what conditions does the Complementation Theorem remain valid when H' is weighted by the *original* weight wt. All arguments in the above proof go through for wt' replaced by wt, except the conclusion of the paragraph following (2.2). More precisely, if we weight H' by wt, it does not follow automatically that the restriction of any $\mu' \in \mathcal{M}_{-A}(H; C_1^-, \ldots, C_l^-)$ to the set of 0-cells has the same weight as the restriction of any $\mu \in \mathcal{M}_A(H; C_1^+, \ldots, C_l^+)$ to the set of 0-cells.

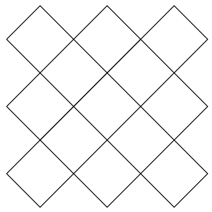
Therefore, the statement of the Complementation Theorem is still valid for H' weighted by the original weight wt, provided that $\mathrm{wt}(\mu_0) = \mathrm{wt}(\mu_0')$ for all μ and μ' as above, where μ_0 denotes the restriction of the matching μ to the set of 0-cells of G (recall that each 0-cell contributes one edge to μ and the opposite edge to μ').

Remark 2.5. Even if the cells of G cannot be partitioned into disjoint lines, when $\Delta(c)$ is constant along each line we can prove that

$$M(H; \mathrm{wt}^2) = M(H'; (\mathrm{wt}')^2) \prod_{L \in \mathcal{L}(G)} \Delta_L^{\tau(L)}, \tag{2.4}$$

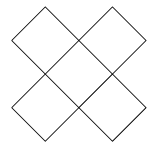
where Δ_L is the value of Δ along L, wt^2 represents the weight obtained by squaring every edge-weight in wt, and $\mathcal{L}(G)$ is the set of lines of G.

Indeed, since each cell is contained in exactly two lines, we have for any matching μ of H that



x_1	y_1	x_2	y_2	x_3	y_3
z_1	w_1	z_2	w_2	z_3	w_3
x_1	y_1	x_2	y_2	x_3	y_3
z_1	w_1	z_2	w_2	z_3	w_3
x_1	y_1	x_2	y_2	x_3	y_3
z_1	w_1	z_2	w_2	z_3	w_3

FIGURE 3.1



x_1	y_2	x_2	y_3
z_1	w_2	z_2	w_3
x_1	y_2	x_2	y_3
z_1	w_2	z_2	w_3

FIGURE 3.2

$$\mathrm{wt}^2(\mu) = \prod_{L \in \mathcal{L}(G)} \mathrm{wt}(\mu|_L),$$

where $\mu|_L$ is the restriction of μ to the cells of L. Using this formula and the arguments in the proof of Theorem 2.1, we deduce that for all $A \in ASP$ we have

$$M_A(H; \mathrm{wt}^2) = M_{-A}(H'; (\mathrm{wt}')^2) \prod_{L \in \mathcal{L}(G)} \Delta_L^{\tau(L)}.$$

Summing over $A \in ASP$ we obtain (2.4).

3. The Aztec diamond

Regard AD_n as a cellular graph and weight the cells in the i-th vertical line by assigning the variables x_i, y_i, w_i and z_i to its four edges, starting with the northwestern edge and going clockwise (the case $n = 3$ is illustrated in Figure 3.1; the array on the right indicates the weight pattern on the edges). Denote the corresponding matching generating function by $M(AD_n; x_1, y_1, z_1, w_1, \dots, x_n, y_n, z_n, w_n)$, or $M(AD_n; x_i, y_i, z_i, w_i)$ for short.

THEOREM 3.1 (STANLEY [**12**]).

$$M(AD_n; x_i, y_i, z_i, w_i) = \prod_{1 \le i \le j \le n} (x_i w_j + z_i y_j). \tag{3.1}$$

Proof. Since cells in the same vertical line are identically weighted, the function Δ is constant along vertical lines. Apply the Complementation Theorem with $G = H = AD_n$ with respect to the partition of the set of cells consisting of vertical lines. The complement H' is the subgraph of AD_n induced by the set of non-extremal vertices, and is clearly isomorphic to AD_{n-1}. According to the definition of the complementary weight, it follows that the four edges in each cell of the i-th vertical line of AD_{n-1} are weighted (clockwise, starting with the northwestern edge) by x_i, y_{i+1}, w_{i+1} and z_i (see Figure 3.2). Therefore, the Complementation Theorem yields that

$$M(AD_n; x_i, y_i, z_i, w_i) = M(AD_{n-1}; x_i, y_{i+1}, z_i, w_{i+1}) \prod_{i=1}^{n} (x_i w_i + z_i y_i). \tag{3.2}$$

To prove formula (3.1), we proceed by induction n. For $n = 1$, the statement is clear. Assuming the formula is true for the diamond of order $n - 1$, we obtain from (3.2) that

$$
\begin{aligned}
M(AD_n; x_i, y_i, z_i, w_i) &= \prod_{i=1}^{n}(x_i w_i + z_i y_i) \prod_{1\le i<j\le n}(x_i w_j + z_i y_j) \\
&= \prod_{1\le i\le j\le n}(x_i w_j + z_i y_j).
\end{aligned}
$$

This completes the induction. □

REMARK 3.2. If we set $x_i = w_i = 1$ and $y_i = z_i = q$ for $i = 1, \ldots, n$, we recover a result of [**5**]. The proof of Theorem 3.1 presented here appears to be new.

REMARK 3.3. The Complementation Theorem can also be used to enumerate recursively the matchings of two other families of graphs, namely, the "halved Aztec diamonds" and the "quasi-quartered Aztec diamonds"; see [**4**].

We conclude this section by presenting a second multivariate version of the Aztec diamond theorem.

Rotate the Aztec diamond so that its edges are horizontal and vertical. Given any matching μ of AD_n, the 0-cells in the ASP associated to μ can have one of the four types N, S, E or W, according to the position of the edge of μ contained in that cell.

Group the cells of AD_n in horizontal (resp., vertical) *ribbons* by collecting cells whose centers have the same y- (resp., x-) coordinate.

LEMMA 3.4. *Consider an arbitrary matching of AD_n. Then in any horizontal ribbon, the number of 0-cells of type E equals the number of 0-cells of type W. A similar statement is true about 0-cells of type N and S in vertical ribbons.*

Proof. The set of matchings of the Aztec diamond is connected under the "elementary move" which for a given matching replaces any two parallel edges contained in a 4-cycle C by the other pair of parallel edges of C (see for example [**16**],[**5**] or [**14**]).

Since the statement of the Lemma is obviously true for the matching consisting entirely of horizontal edges, it suffices to check that in any (say horizontal) ribbon, the above-described elementary move preserves the difference between the number of 0-cells of type E and those of type W.

If the 4-cycle C to which we apply the move is a cell, the 0-cells are not affected by the move, so we have nothing to check. Suppose therefore that C is not a cell. Consider first the case when the two edges of μ contained in C are vertical. Let R be the horizontal ribbon containing C.

Then we have, up to symmetry, three different possibilities on the cells of R adjacent to C: they are either both 0-cells, or both 1-cells, or one of them is a 0-cell and the other a 1-cell. The three cases are illustrated in Figure 3.3, which also shows how the matching is affected after the elementary move. It is readily seen that the difference between the number of 0-cells of type E and W contained in R is not affected by the move.

The case when the two edges of the original matching contained in C are horizontal follows by reading Figure 3.3 from right to left. □

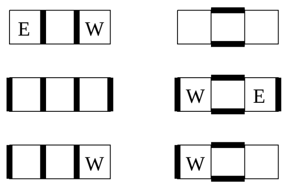


FIGURE 3.3

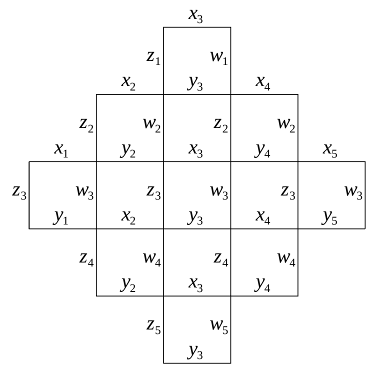


FIGURE 3.4

Define a weight wt on AD_n as follows. Weight the horizontal edges in the i-th vertical ribbon (counting from left) alternately by x_i and y_i (see Figure 3.4). Weight the vertical edges in the i-th horizontal ribbon (counting from top) alternately by z_i and w_i ($i = 1, \ldots, 2n-1$). Denote by

$$M(AD_n; \mathrm{wt}) = M(AD_n; x_1, y_1, \ldots, x_{2n-1}, y_{2n-1}; z_1, w_1, \ldots, z_{2n-1}, w_{2n-1})$$

the matching generating function of AD_n with respect to this weight.

THEOREM 3.5. *Suppose there exists a constant c such that $x_{i+1}y_{i+1} - x_iy_i = z_{i+1}w_{i+1} - z_iw_i = c$, for $i = 1, \ldots, 2n-2$. Then*

$$M(AD_n, \mathrm{wt}) = \prod_{1 \leq i \leq j \leq n} (x_jy_j + z_{n-i+j}w_{n-i+j}) = \prod_{1 \leq i \leq j \leq n} (a + (n - i + 2j)c), \tag{3.3}$$

where $a = x_1y_1 + z_1w_1 - 2c$.

We note the resemblance of the above formula to Theorem 3.1.

Proof. By Lemma 3.4, the condition mentioned in Remark 2.4 is met in our case. Also, the equality of the specified $4n-4$ numbers implies that the function Δ is constant both along the lines of AD_n of slope 1 and along the lines of AD'_n of slope 1. Since AD'_n is isomorphic to AD_{n-1}, the variant of the Complementation Theorem mentioned in Remark 2.4 can be applied repeatedly.

To prove the first equality in (3.3), we proceed by induction on n. For n=1 the formula is clearly true. Suppose it is true for the diamond of order $n-1$. Apply the variant of the Complementation Theorem presented in Remark 2.4 with both H and G chosen to be AD_n, with respect to the lines of slope 1. We obtain (see Figure 3.4) that

$$\begin{aligned} M(AD_n; x_1, y_1, \ldots, &x_{2n-1}, y_{2n-1}; z_1, w_1, \ldots, z_{2n-1}, w_{2n-1}) \\ &= \prod_{j=1}^{n} (x_jy_j + z_{n+j-1}w_{n+j-1}) \\ &\quad \cdot M(AD_{n-1}; y_2, x_2, \ldots, y_{2n-2}, x_{2n-2}; w_2, z_2, \ldots, w_{2n-2}, z_{2n-2}). \end{aligned}$$

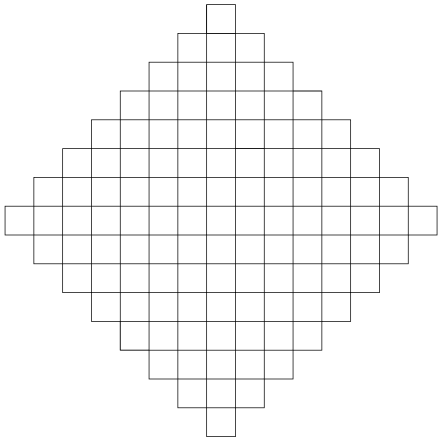


FIGURE 4.1

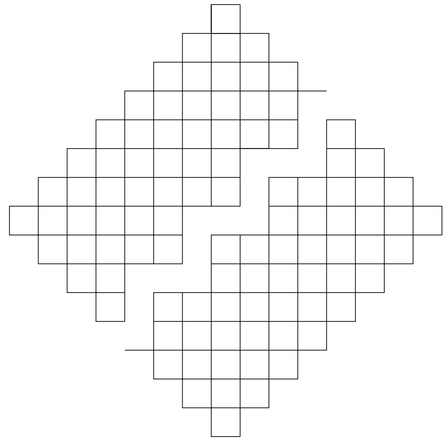


FIGURE 4.2

Using the induction hypothesis, the right hand side of the above relation can be successively rewritten as

$$\prod_{j=1}^{n}(x_jy_j + z_{n+j-1}w_{n+j-1}) \prod_{1\le i\le j\le n-1} (x_{j+1}y_{j+1} + z_{(n-1)-i+j+1}w_{(n-1)-i+j+1})$$
$$= \prod_{j=1}^{n}(x_jy_j + z_{n+j-1}w_{n+j-1}) \prod_{2\le i\le j\le n} (x_jy_j + z_{n-i+j}w_{n-i+j})$$
$$= \prod_{1\le i\le j\le n} (x_jy_j + z_{n-i+j}w_{n-i+j}),$$

thus completing the proof of the first equality in (3.3) by induction. The second equality follows immediately using the fact that the $4n-4$ numbers in the statement of the Theorem are equal to c. $\square$

4. Quartered Aztec diamonds

In [**6**] there are introduced three families of graphs called *quartered Aztec diamonds* which can be described as follows.

Consider the Aztec diamond of order n (Figure 4.1 illustrates the case $n = 8$). We can divide this graph into two congruent parts by deleting edges in a zig-zag pattern, changing direction every two steps (see Figure 4.2). By superimposing two such "deletion" paths that intersect at the center of the graph we divide it into four pieces; the resulting subgraphs are called *quartered Aztec diamonds* (technically speaking, these are the dual graphs of the planar regions called quartered Aztec diamonds in [**6**]).

Up to symmetry, there are two different ways we can superimpose the two deletion paths. For one of them, the obtained pattern has fourfold rotational symmetry and the four subgraphs are isomorphic (see Figure 4.3(a)); we call them rotationally quartered Aztec diamonds. For the other, the resulting pattern has Klein 4-group reflection symmetry and

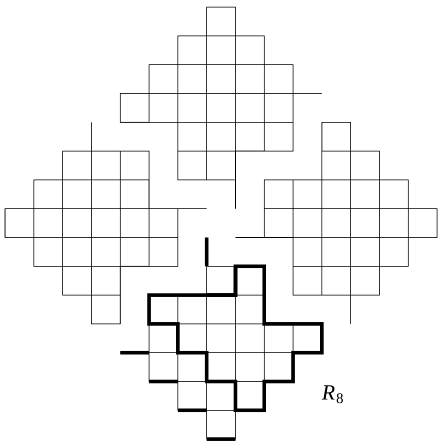


FIGURE 4.3(a)

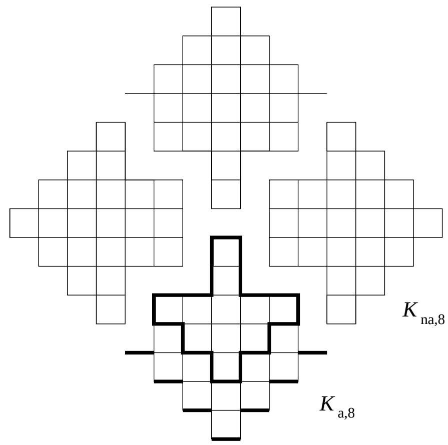


FIGURE 4.3(b)

there are two different kinds of subgraphs (see Figure 4.3(b)); they are called "abutting" and "non-abutting" quartered Aztec diamonds.

In [**6**] there are obtained product formulas for the number of matchings of all three kinds of quartered Aztec diamonds. Once these formulas are derived, it is noticed that various of the obtained numbers differ only by a multiplicative factor of a power of two (see relations (1)–(4) of [**6**]). Direct proofs, which are combinatorial to some extent, are given in the same paper for relations (1), (3) and (4). We present a unified combinatorial proof for all four relations based on the Complementation Theorem. This appears to be the first direct proof of (2).

Consider the three types of quartered Aztec diamonds of order n. Delete all vertices on which perfect matchings are forced (see Figures 4.3(a) and (b)). Denote by R_n, $K_{\mathrm{a},n}$ and $K_{\mathrm{na},n}$ the graphs obtained this way from the rotationally quartered, abutting and non-abutting quartered Aztec diamond of order n, respectively. The four relations in the statement of the following Theorem are equivalent to equations (1)–(4) of [**6**].

THEOREM 4.1. *For all $n \geq 1$ we have*

$$\begin{aligned} M(R_{4n}) &= 2^n M(R_{4n-1}) && (4.1)\\ M(K_{\mathrm{na},4n+1}) &= 2^n M(K_{\mathrm{na},4n}) && (4.2)\\ M(K_{\mathrm{na},4n}) &= 2^n M(K_{\mathrm{a},4n-1}) && (4.3)\\ M(K_{\mathrm{a},4n-2}) &= 2^n M(K_{\mathrm{na},4n-3}). && (4.4) \end{aligned}$$

In our proof we will employ the following result.

LEMMA 4.2. *Let G be a weighted graph having a 7-vertex subgraph H consisting of two 4-cycles that share a vertex. Let a, b_1, b_2, b_3 and a, c_1, c_2, c_3 be the vertices of the 4-cycles (listed in cyclic order) and suppose b_3 and c_3 are the only vertices of H with neighbors outside H. Let $\bar{G}$ be the subgraph of G obtained by deleting b_1, b_2, c_1 and c_2, weighted*

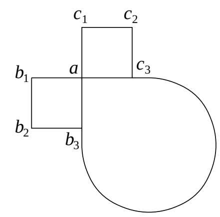


FIGURE 4.4

by restriction. Then if the product of weights of opposite edges in each 4-cycle of H is constant, we have

$$M(G) = 2\,\mathrm{wt}(b_1, b_2)\,\mathrm{wt}(c_1, c_2)M(\bar{G}).$$

Proof. The set of matchings of G can be partitioned in three classes (see Figure 4.4): matchings containing $\{b_1, b_2\}$, $\{c_1, c_2\}$; matchings containing $\{b_1, b_2\}$, $\{c_1, a\}$, $\{c_2, c_3\}$; and matchings containing $\{c_1, c_2\}$, $\{b_1, a\}$, $\{b_2, b_3\}$.

The matchings of the first class are clearly in bijection with the matchings of $\bar{G}$. Their contribution to $M(G)$ is $\mathrm{wt}(b_1, b_2)\,\mathrm{wt}(c_1, c_2)M(\bar{G})$.

The matchings of the second class are in bijection with the matchings of $\bar{G}$ in which a is matched to c_3. Indeed, let μ be a matching of G in the second class. The operation of removing $\{b_1, b_2\}$ and replacing the two edges $\{c_1, a\}$ and $\{c_2, c_3\}$ by $\{a, c_3\}$ gives the desired bijection. Thus the contribution of the second class to $M(G)$ is

$$\mathrm{wt}(b_1, b_2)\,\mathrm{wt}(c_1, a)\,\mathrm{wt}(c_2, c_3)\,\mathrm{wt}(a, c_3)^{-1}M_1(\bar{G}) = \mathrm{wt}(b_1, b_2)\,\mathrm{wt}(c_1, c_2)M_1(\bar{G}),$$

where $M_1(\bar{G})$ is the generating function for matchings of $\bar{G}$ with a matched to c_3 (we used here our hypothesis about the products of weights of opposite edges in the 4-cycles of H).

The same argument shows that the contribution of the matchings of G in the third class to $M(G)$ is $\mathrm{wt}(b_1, b_2)\,\mathrm{wt}(c_1, c_2)M_2(\bar{G})$, where $M_2(\bar{G})$ is the generating function for matchings of $\bar{G}$ with a matched to b_3.

Since the G-neighborhood of a is contained in H, the $\bar{G}$-neighborhood of a consists of b_3 and c_3. Therefore, the matchings in the second and third class contribute jointly to $M(G)$ the same amount as the matchings of the first class. We therefore obtain the statement of the Lemma. □

Let H be a finite connected induced subgraph of the grid graph $\mathbf{Z}^2$; suppose H has no vertices of degree one (from the point of view of matching enumeration we can make the latter assumption without loss of generality). From the set of 4-faces with (centers having) minimum x-coordinate, choose the 4-face c with maximum y-coordinate. Then H has two cellular completions: in one of them c is a cell (call this the *even* cellular completion); in the other it is not (call this the *odd* cellular completion). An example is shown in Figure 4.5.

Proof of Theorem 4.1. Consider the graphs appearing on the left hand side of equations (4.1)–(4.4) and position them so that the corner of the Aztec diamond from which they were obtained is pointing downward (see Figure 4.6).

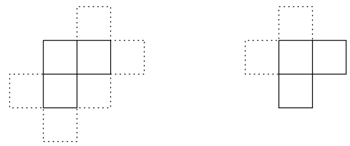
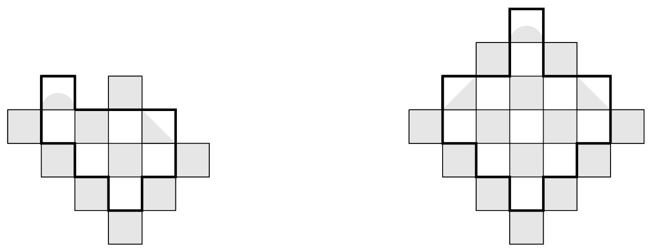


FIGURE 4.5



(a). Obtaining R_7 from R_8. (b). Obtaining $K_{\mathrm{a},7}$ from $K_{\mathrm{na},8}$.

To prove (4.1), apply the Complementation Theorem to R_{4n} (with unit weights on the edges) with respect to its even cellular completion. This is illustrated in Figure 4.6(a) for $n = 2$. (The cellular completion is the union of 4-cycles whose interiors are shaded (totally or partially); the edges of the original graph are precisely those contained in the shaded region).

However, the complement of R_{4n} is isomorphic to R_{4n-1} (its boundary is pictured in thick solid lines in Figure 4.6(a)). Since in R_{4n} the only lines of slope 1 having nonzero type are n lines of type 1, we obtain (4.1).

Similarly, by applying the Complementation Theorem to $K_{\mathrm{na},4n}$ and $K_{\mathrm{a},4n-2}$ with respect to their even cellular completion, one obtains (4.3) and (4.4), respectively (see Figures 4.6(b) and (c)).

To prove formula (4.2), apply the Complementation Theorem to $K_{\mathrm{na},4n+1}$ with respect to its odd cellular completion (see Figure 4.6(d)). Consider the partition of this cellular completion into lines of slope 1. Since the only lines of nonzero type are n lines of type -1, we obtain

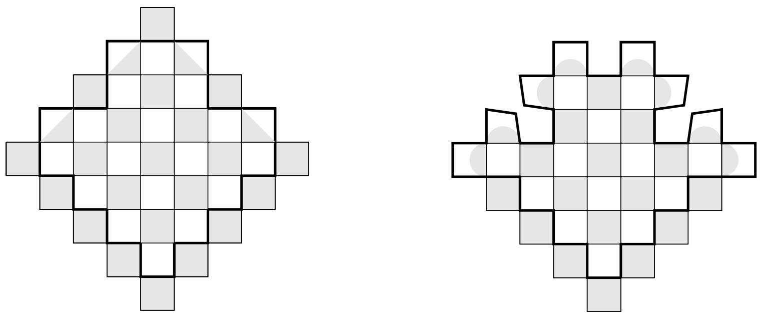
$$M(K_{\mathrm{na},4n+1}) = 2^{-n} M(K'_{\mathrm{na},4n+1}). \tag{4.5}$$

Furthermore, in the graph $K'_{\mathrm{na},4n+1}$ we have $2n$ opportunities to apply Lemma 4.2 (see Figure 4.6(d)). However, the graph obtained from $K'_{\mathrm{na},4n+1}$ after deleting the vertices prescribed by Lemma 4.2 is isomorphic to $K_{\mathrm{na},4n}$. Thus relation (4.5) implies (4.2). □

The *holey Aztec diamond* is the graph obtained by removing the vertices of the central 4-cycle of the Aztec diamond (which we consider centered at the origin and placed so that its lines of cells are parallel to the coordinate axes; see Figure 4.7). Let r denote the rotation by 90° around the origin.

The following result is used in [**3**] to deduce a certain weighted enumeration of quarter-turn invariant alternating sign matrices of order $4k - 1$.

THEOREM 4.2. *The holey Aztec diamond of order* $4k-2$ *has* $M(R_{4k-1})$ *r-invariant perfect matchings.*



(c). Obtaining $K_{\mathrm{na},9}$ from $K_{\mathrm{a},10}$. (d). Obtaining $K_{\mathrm{na},8}$ from $K_{\mathrm{na},9}$.

Figure 4.6

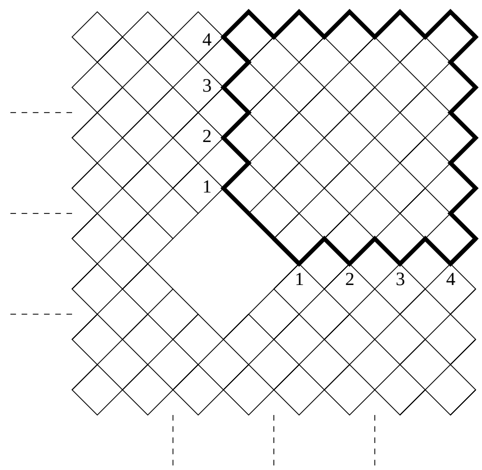


FIGURE 4.7

Proof. Let G be the subgraph of the holey Aztec diamond induced by the vertices in the (closed) first quadrant (see Figure 4.7; the boundary of G is shown in boldface). Label the vertices of G on the coordinate axes as follows: label by 1 the two vertices closest to the origin, by 2 the two next closest and so on, ending with two vertices labeled $2k-2$. Then the r-invariant matchings of the holey Aztec diamond can be identified with the matchings of the graph $\tilde{G}$ obtained from G by superimposing vertices with the same label.

The graph $\tilde{G}$ can be embedded in the plane so that it is a "symmetric graph" in the sense of [**3**] (see Figure 4.8(a)). Then the argument in the proof of Theorem 7.1 of [**3**] shows that

$$M(\tilde{G}) = 2^{k-1} M(\tilde{G}_1), \tag{4.6}$$

where $\tilde{G}_1$ is the graph obtained from $\tilde{G}$ by removing the dotted edges in Figure 4.8(a). Another plane embedding of $\tilde{G}_1$ is shown in Figure 4.8(b). Remove the vertices on which

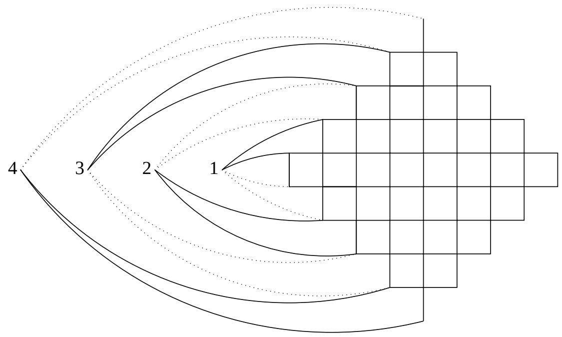


FIGURE 4.8(a)

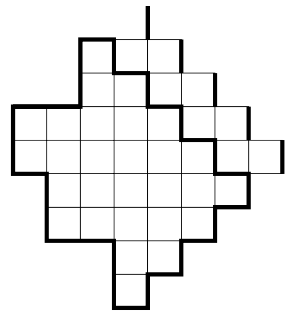


FIGURE 4.8(b)

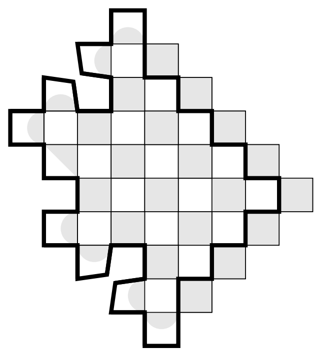


FIGURE 4.9

perfect matchings are forced; denote the resulting graph by $\tilde{G}_2$.

Consider now the graph R_{4k-1} (positioned as the piece on the right of Figure 4.3(a)) and apply the Complementation Theorem with respect to its odd cellular completion (see Figure 4.9). Since the only lines of slope 1 having nonzero type are $k-1$ lines of type -1, this yields a multiplicative factor of $2^{-(k-1)}$. On the other hand, the complement R'_{4k-1} provides $2k-2$ opportunities to apply Lemma 4.2, thus producing a multiplicative factor of 2^{2k-2}. Moreover, the resulting graph is just $\tilde{G}_2$. Therefore we obtain

$$M(R_{4k-1}) = 2^{k-1} M(\tilde{G}_2). \tag{4.7}$$

Since $\tilde{G}_1$ and $\tilde{G}_2$ clearly have the same number of matchings, the statement of the Theorem follows from (4.6) and (4.7). □

5. Spanning trees

A natural generalization of the Aztec diamonds is the following. Consider a $(2m+1) \times (2n+1)$ chessboard with black corners. The graph whose vertices are the white squares and whose edges connect precisely those pairs of squares that are diagonally adjacent is called the *(even) Aztec rectangle* of order $m \times n$, and is denoted $ER_{m,n}$ (we note that this definition of the Aztec diamond differs from the one in [**2**]). The analogous graph constructed on the black squares is called the *odd Aztec rectangle* of order $m \times n$, and is

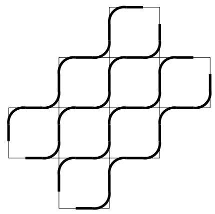


FIGURE 5.1(a)

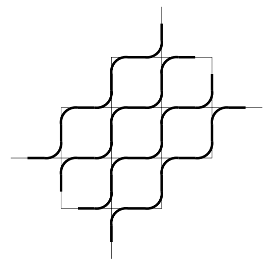


FIGURE 5.1(b)

denoted $OR_{m,n}$. Note that $ER_{n,n}$ is just the Aztec diamond AD_n; accordingly, $OR_{n,n}$ is called the *odd Aztec diamond* of order n and is denoted OD_n.

Stanley conjectured [**13**] that the number of spanning trees of AD_n is precisely 4 times the number of spanning trees of OD_n. This has been later proved by Knuth [**7**], who showed that in fact the result is true in the more general case of Aztec rectangles.

We present a weighted version of this result, which proves a particular case of the recently solved more general conjecture of Chow [**1**]. In contrast to the proofs of Knuth and Chow, which rely on linear algebra, our argument is combinatorial.

We will find it convenient to rotate the Aztec rectangles by 45° so that their edges are horizontal and vertical. Define weights on the edges of the even and odd Aztec rectangles as follows. Partition the edges in both $m \times n$ Aztec rectangles into staircase-like paths that take alternating steps north and east, as indicated in Figure 5.1 for $m = 2$,$n = 3$. Weight the edges contained in the i-th path from top by x_i, for $i = 1, \dots, 2m$.

The weight of a spanning tree T of the weighted graph G is defined to be the product of the weights of the edges of T. The spanning tree generating function of G, denoted $t(G)$, is the total weight of the spanning trees of G.

THEOREM 5.1. *For the above-described weight on the even and odd Aztec rectangles, we have*

$$t(ER_{m,n}) = \frac{2}{x_1} \prod_{i=1}^{m} \frac{x_{2i-1} + x_{2i}}{x_{2i} + x_{2i+1}} t(OR_{m,n}),$$

where by definition $x_{2m+1} = 0$.

Our argument employs the following result due to Temperley (see [**15**] and Exercise 4.30 of [**10**]).

Let G be a finite connected weighted subgraph of the grid $\mathbf{Z}^2$ such that all finite faces are unit squares. Color the vertices of G black. Divide each edge of G in two by inserting green vertices at their midpoints; weight both newly formed edges by the weight of the original edge. Divide each face of G in four by inserting a red vertex at its center and joining it to the green vertices on its boundary by edges of weight 1. Let $\tilde{G}$ be the graph on the black, green and red vertices obtained in this fashion.

LEMMA 5.2. *If v is a black vertex on the boundary of $\tilde{G}$, then there is a weight-preserving bijection between the spanning trees of G and the perfect matchings of $\tilde{G} \setminus \{v\}$.*

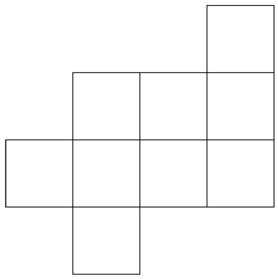


FIGURE 5.2(a)

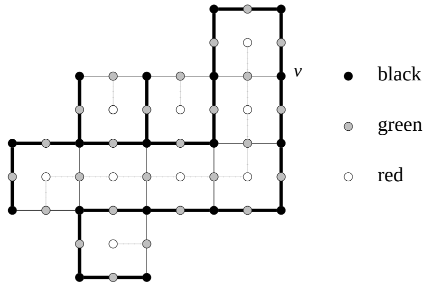


FIGURE 5.2(b)

Proof. Regard G as being the graph on the black vertices of $\tilde{G}$; let T be a spanning tree of G (see Figures 5.2(a) and (b)). For any black vertex $x \neq v$, let x' be the first green vertex encountered along the unique path joining x to v in T.

Next, note that if we include an extra red vertex u for the infinite face of G (adjacent to the green vertices on the edges of the boundary of G), the red vertices are the vertices of a spanning tree T^* (dual to T) of the dual graph of G. For any red vertex $y \neq u$, let y' be the first green vertex encountered along the unique path joining y to u in T^*.

The collection μ_T consisting of the edges $\{x, x'\}$, $\{y, y'\}$ with x (resp., y) running over black vertices different from v (resp., red vertices different from u) is clearly a perfect matching of $\tilde{G} \setminus \{v\}$. Furthermore, the weight of μ_T is equal to the weight of T.

Conversely, let μ be a perfect matching of $\tilde{G} \setminus \{v\}$. Let T_μ be the subgraph of G formed by the edges containing some member of μ. Since the members of μ contained in edges of G are precisely those incident to black vertices, T_μ has $V(G) - 1$ edges.

To show that T_μ is a spanning tree of G it is therefore enough to prove that T_μ contains no cycle. Suppose this is not the case and let C be a cycle without self-intersections. By induction on the length of C we see that the number of vertices of $\tilde{G}$ encircled by C is odd. Since the removed vertex v belongs to the boundary of the infinite face, it follows that C encircles an odd number of vertices of $\tilde{G} \setminus \{v\}$. However, these cannot be matched by μ, a contradiction. Therefore, T_μ is a spanning tree of G, and its weight is clearly equal to the weight of μ.

Since the maps $T \mapsto \mu_T$ and $\mu \mapsto T_\mu$ are inverse to one another, we obtain the statement of the Lemma. $\square$

The following variation of the above lemma will also be relevant in our proof.

Let G be as in the statement of Lemma 5.2; let wt_G be its weight function. Consider the partition of the edges of the grid $\mathbf{Z}^2$ into staircase-like paths taking alternating steps north and east. Weight the edges of G contained in such a staircase P by x_P.

Let $\tilde{G}_1$ be the graph obtained from $\tilde{G}$ by "reshuffling" the weights as follows. Give each edge connecting a red vertex r to a green vertex g the weight of the edge of G contained in the 4-face centered at r and opposite the edge of G containing g (see Figure 5.3). Give weight 1 to all edges connecting a black vertex to a green vertex.

Let v be a black vertex on the boundary of $\tilde{G}_1$. Let φ be the bijection of Lemma 5.2 between the spanning trees of G and the perfect matchings of $\tilde{G}_1 \setminus \{v\}$. Denote by $\mathrm{wt}_{\tilde{G}_1 \setminus \{v\}}$ the weight on $\tilde{G}_1 \setminus \{v\}$ obtained by restriction from the weight of $\tilde{G}_1$ defined in the previous

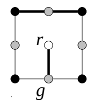


FIGURE 5.3

paragraph.

It turns out that even though we have reshuffled the weights on $\tilde{G}$, φ is still weight-preserving, up to a multiplicative factor.

LEMMA 5.3. *The quotient* $\operatorname{wt}_{\tilde{G}_1\setminus\{v\}}(\varphi(T))/\operatorname{wt}_G(T)$ *is the same for all spanning trees* T *of* G.

Proof. The set of spanning trees of a graph is connected under the operation consisting of removing an edge of a spanning tree and including some other edge of the graph so as to form another spanning tree. Therefore, it suffices to show that if T' was obtained from T by such an operation, then

$$\operatorname{wt}_{\tilde{G}_1\setminus\{v\}}(\varphi(T'))/\operatorname{wt}_G(T') = \operatorname{wt}_{\tilde{G}_1\setminus\{v\}}(\varphi(T))/\operatorname{wt}_G(T). \tag{5.1}$$

The edges in the matching $\varphi(T)$ are of two types: edges contained in T, which we will call of type I, and edges lying along T^*, which we will call of type II. By our definition, $\operatorname{wt}_{\tilde{G}_1\setminus\{v\}}(\varphi(T))$ is determined by the edges of $\varphi(T)$ of type II.

Therefore, to compare $\operatorname{wt}_{\tilde{G}_1\setminus\{v\}}(\varphi(T))$ and $\operatorname{wt}_{\tilde{G}_1\setminus\{v\}}(\varphi(T'))$ it is enough to determine how they differ on the edges of type II.

Let $\{e\} = T \setminus T'$, $\{f\} = T' \setminus T$. Then T'^* is obtained from T^* by including e^* and removing f^* (e^* is the edge of G^* dual to e). The inclusion of e^* creates a cycle C^*; f^* must lie along this cycle. Note also that C^* contains the vertex ∞ corresponding to the infinite face of G (indeed, the inclusion of e^* in T^* corresponds to the removal of e from T; the remaining subgraph of T consists of two disjoint trees, and C^* corresponds to a contiguous sequence of faces of G separating the two trees).

By the definition of φ it follows that the edges of type II in $\varphi(T)$ and $\varphi(T')$ differ only along the path P of C^* which connects e^* to f^*, avoiding ∞. We can further specify this difference as follows.

Note that each edge of the dual graph G^* not containing ∞ accounts for two edges in $\tilde{G}_1 \setminus \{v\}$. Therefore, we can regard P as a path in $\tilde{G}_1 \setminus \{v\}$ connecting the midpoint of e to the midpoint of f. It follows that there are an even number of edges of $\tilde{G}_1 \setminus \{v\}$ in P.

The difference between the edges of type II in $\varphi(T)$ and $\varphi(T')$ is then the following: $\varphi(T)$ contains the alternating set of edges of P matching the ending point of P, while $\varphi(T')$ contains the alternating set of edges of P matching the starting point of P.

We can now restate relation (5.1) as follows. Consider the graph $U = \mathbf{Z}^2$ and divide each edge in two by inserting a new vertex at its midpoint. Denote the resulting graph by S.

Partition the edges of the dual graph U^* into staircase-like paths taking alternately steps north and east (see Figure 5.4(a)). For each edge e^* of U^* let M_{e^*} be the set consisting of the two edges of S perpendicular to e^* and with endpoints at distance $1/2$ and 1 from

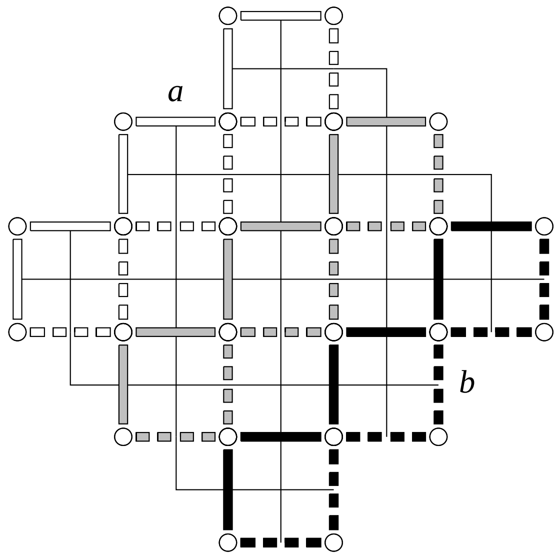


FIGURE 5.4(a)

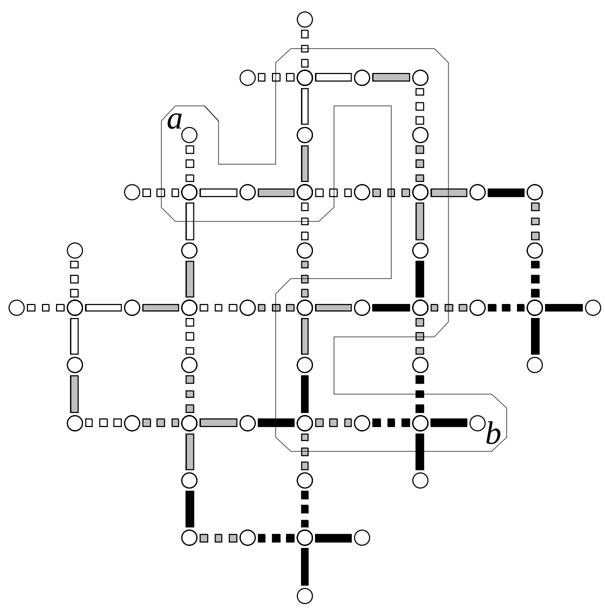


FIGURE 5.4(b)

the midpoint of e^*. For each staircase P in U^* let each edge in $\bigcup_{e^* \in P} M_{e^*}$ have weight x_P (see Figure 5.4(b)).

To prove (5.1), it suffices to show that for any path of S having (distinct) edges $e_1, \ldots, e_{2n}$ connecting two vertices $a, b \in V(S) \setminus V(U)$, we have

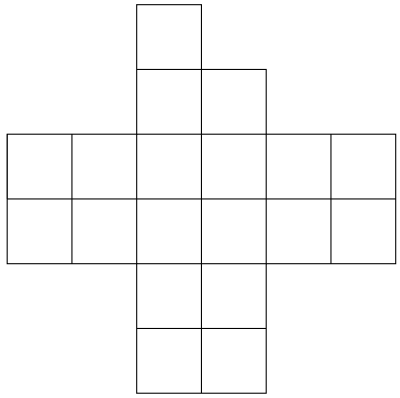


FIGURE 5.5

$$\frac{\mathrm{wt}(e_1)\,\mathrm{wt}(e_3)\cdots\mathrm{wt}(e_{2n-1})}{\mathrm{wt}(e_2)\,\mathrm{wt}(e_4)\cdots\mathrm{wt}(e_{2n})}=\frac{x_Q}{x_P}, \tag{5.2}$$

where P and Q are the staircases of U^* containing a and b, respectively. Indeed, as we have seen above, the ratio between $\mathrm{wt}_{\tilde{G}_1\setminus\{v\}}(\varphi(T'))$ and $\mathrm{wt}_{\tilde{G}_1\setminus\{v\}}(\varphi(T))$ can be written in the form of the left hand side of (5.2) (in this case, a and b are the midpoints of e and f, respectively). On the other hand, since T' was obtained from T by removing e and including f, $\mathrm{wt}_G(T')/\,\mathrm{wt}_G(T)$ equals the right hand side of (5.2).

However, $V(S)\setminus V(U)$, the staircases of U^* and the weight on S are all invariant under reflections in lines of slope -1 through the vertices of U. Since by means of such reflections we can render straight our path connecting a to b, it suffices to prove (5.2) in the case when all the edges in this path are horizontal. As the sequence of edge-weights $(\mathrm{wt}(e_i))$ in this case is of type

$$\ldots, x_1, x_{-1}, x_2, x_1, x_3, x_2, x_4, x_3, x_5, x_4, x_6, x_5, x_7, x_6 \ldots,$$

and since $x_Q=\mathrm{wt}(e_{2n-1})$, $x_P=\mathrm{wt}(e_2)$, relation (5.2) is clearly true in this case. $\square$

Proof of Theorem 5.1. Let $E=E_{m,n}$ be the graph obtained from $ER_{m,n}$ after applying the construction in Lemma 5.2, with the deleted vertex chosen to be the upper endpoint of the topmost line of slope 1 in $ER_{m,n}$ (for $m=n=2$, this is illustrated in Figure 5.5). It suffices show that

$$M(E)=2\frac{\prod_{i=1}^{m}(x_{2i-1}+x_{2i})}{x_1x_{2m}\prod_{i=1}^{m-1}(x_{2i}+x_{2i+1})}t(OR_{m,n}). \tag{5.3}$$

Consider the even cellular completion of E (see Figure 5.6(a)). Note that the function Δ is constant along its lines of slope 1: it equals $2x_i$ along the i-th line from top. Also, the first two lines from top have type zero, and all others have type -1. Denoting by E' the

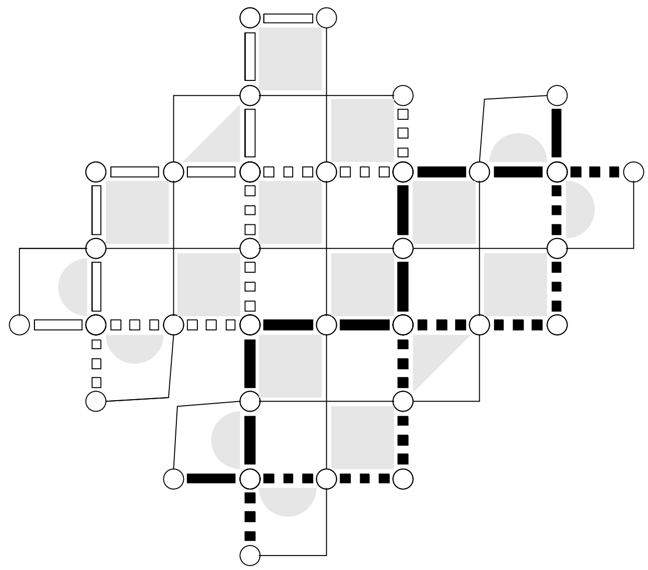


FIGURE 5.6(a)

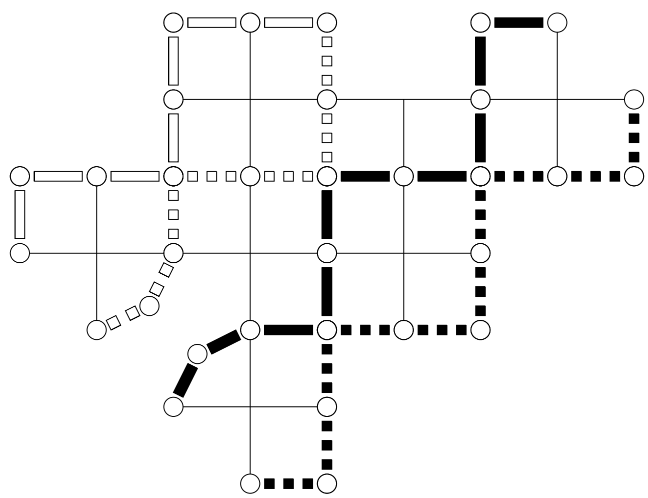


FIGURE 5.6(b)

complement of E with respect to the chosen cellular completion, the Complementation Theorem yields

$$M(E) = (2x_1)^0(2x_2)^0(2x_3)^{-1}\cdots(2x_{2m})^{-1}M(E'). \tag{5.4}$$

In the graph E' we have $2m-1$ opportunities to apply Lemma 4.2 (see Figure 5.6(b)). Let F be the graph obtained from E' after deleting the $2m-1$ quadruples of vertices prescribed in the statement of Lemma 4.2 (F, together with its cellular completion, is

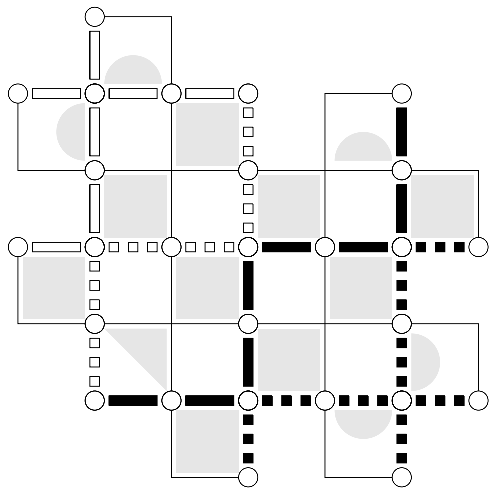


FIGURE 5.6(c)

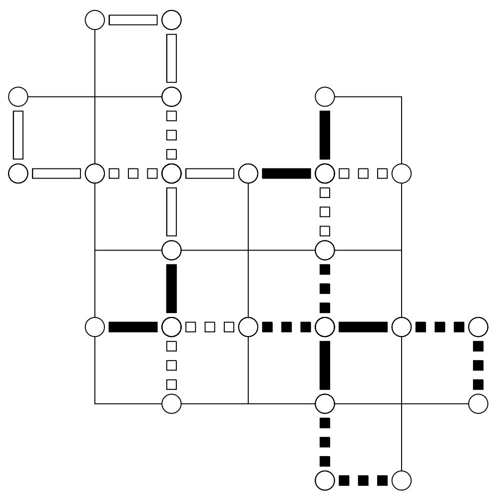


FIGURE 5.6(d)

pictured in Figure 5.6(c)). We obtain

$$M(E') = (2x_1x_2)(2x_3x_4)^2 \cdots (2x_{2m-1}x_{2m})^2 M(F). \tag{5.5}$$

Next, apply the Complementation Theorem to F and its even cellular completion. In each cell of the top $n-1$ lines of slope 1 of this cellular completion, assign to the edges not contained in F weights 1 and x_1 such that the weight pattern is symmetric with respect to a straight line of slope 1 (see Figure 5.6(c)). Weight the edges in the bottom $n-1$ lines not contained in F in a similar fashion, by 1's and x_{2m}'s. Then, starting from the top, the lines of slope 1 of this cellular completion are as follows: $n-1$ lines of type -1 on which Δ takes value $2x_1$; $2m-1$ lines whose types are alternately 1 and -1, with the value of Δ

equal to $(x_i + x_{i+1})$ on the i-th of them; and $n-1$ lines of type -1 along which Δ equals $2x_{2m}$.

Therefore, the Complementation Theorem gives

$$M(F) = (2x_1)^{-(n-1)}(x_1+x_2)(x_2+x_3)^{-1}(x_3+x_4)\cdots(x_{2m-1}+x_{2m}) \cdot (2x_{2m})^{-(n-1)}M(F'). \tag{5.6}$$

In the graph F' we have $2(n-1)$ occasions to apply Lemma 4.2 (see Figure 5.6(d)). Let H be the resulting graph. We obtain

$$M(F') = (2x_1^2)^{n-1}(2x_{2m}^2)^{n-1}M(H). \tag{5.7}$$

Relations (5.4)–(5.7) imply

$$M(E) = 2x_1^n x_2 x_3 \cdots x_{2m-1} x_{2m}^n \frac{(x_1+x_2)(x_3+x_4)\cdots(x_{2m-1}+x_{2m})}{(x_2+x_3)(x_4+x_5)\cdots(x_{2m-2}+x_{2m-1})} M(H). \tag{5.8}$$

However, H is precisely the graph obtained from $OR_{m,n}$ by removing the four leaves and then applying the construction involved in Lemma 5.3. We deduce by this Lemma that $M(H)$ and $t(OR_{m,n})$ differ only by a multiplicative factor. This factor can be determined for example by choosing T to be the spanning tree of $OD_{m,n}$ consisting of all its horizontal edges together with all "leftmost" vertical edges. We obtain

$$M(H) = x_1^{-(n+1)} x_2^{-1} x_3^{-1} \cdots x_{2m-1}^{-1} x_{2m}^{-(n+1)} t(OR_{m,n}),$$

which combined with (5.8) proves relation (5.3) and hence the statement of the theorem. □

6. Alternating sign matrices

The ASP's having the shape of an Aztec diamond are known as *alternating sign matrices.* The problem of determining the number of alternating sign matrices of a given order turned out to be among the hardest in enumerative combinatorics. In 1983, Mills, Robbins and Rumsey [**11**] conjectured that this number is given by a certain simple product formula. This was first proved by Zeilberger [**18**] and later by Kuperberg [**8**].

Let $ASM(n)$ be the set of alternating sign matrices of order n. Weight each $A \in ASM(n)$ by $x^{N_-(A)}$ and let $A_n(x)$ be their generating function. It turns out that for $x = 2$ we can determine various refinements of $A_n(x)$ using the Complementation Theorem.

Indeed, taking all weights in Theorem 3.1 to be 1 we obtain $A_n(2) = 2^{\binom{n}{2}}$. Furthermore, let $A_{n,k}(x)$ be the generating function of those matrices in $ASM(n)$ whose unique 1 in the first row is in position k. It is proved in [**11**] that $A_{n,k}(2) = 2^{\binom{n-1}{2}}\binom{n-1}{k-1}$ (see [**5**] for an alternative proof); this also follows from the more general result presented below.

Refining further the generating function under consideration, let $A_{n,k,h}(x)$ be the generating function of alternating sign matrices of order n in which the unique 1's in the first and last row occupy positions k and h, respectively ($1 \le k, h \le n$).

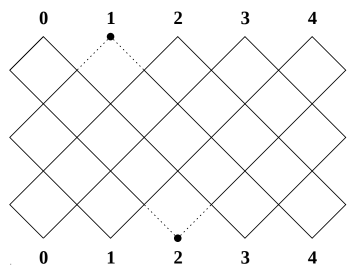


FIGURE 6.1

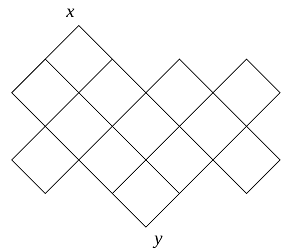


FIGURE 6.2

THEOREM 6.1. *For $n \geq 2$ we have*

$$A_{n,k,h}(2) = 2^{\binom{n-2}{2}} \left(\binom{n-2}{k-1}\binom{n-2}{h-2} + \binom{n-2}{k-2}\binom{n-2}{h-1} \right).$$

Proof. For $n = 2$ the formula is clearly true. Suppose therefore $n \geq 3$. Consider the cellular graph consisting of the middle $n-2$ horizontal lines of cells of AD_n. Label its top and bottom n vertices consecutively from left to right by $0, 1, \ldots, n-1$. Let $H_{n-1,k,h}$ be the subgraph obtained by deleting the vertex labeled k in the top row and the one labeled h in the bottom row (an example is shown in Figure 6.1). Via the correspondence between ASM's of order n and matchings of AD_n described in the proof of the Complementation Theorem, we obtain that

$$A_{n,k,h}(2) = 2^{2-n} M(H_{n-1,k-1,h-1}). \tag{6.1}$$

Now apply the Complementation Theorem to $H_{n,k,h}$ with respect to the cellular completion consisting of the middle $n-1$ horizontal lines of cells of AD_{n+1}. Let x and y be the highest and lowest vertices of the complement of $H_{n,k,h}$ (see Figure 6.2). Since they both have degree two, the matchings of $H'_{n,k,h}$ are divided in four classes according to the position of the edges matching x and y. However, each of these four classes is in bijection with the matchings of some graph $H_{n-1,i,j}$. More precisely, we obtain

$$\begin{aligned} M(H_{n,k,h}) = 2^{n-1}(M(H_{n-1,k-1,h-1}) &+ M(H_{n-1,k-1,h}) \\ &+ M(H_{n-1,k,h-1}) + M(H_{n-1,k,h})) \end{aligned}$$

(in case k or h is out of the range $\{0, \ldots, n\}$, $M(H_{n,k,h})$ is taken to be zero.)

If we define $g_{n,k,h} := M(H_{n,k,h})/2^{\binom{n}{2}}$, we can rewrite the last equation as

$$g_{n,k,h} = g_{n-1,k-1,h-1} + g_{n-1,k-1,h} + g_{n-1,k,h-1} + g_{n-1,k,h}. \tag{6.2}$$

Consider the numbers $a_{n,k,h}$ given by

$$a_{n,k,h} := \binom{n-1}{k}\binom{n-1}{h-1} + \binom{n-1}{k-1}\binom{n-1}{h}.$$

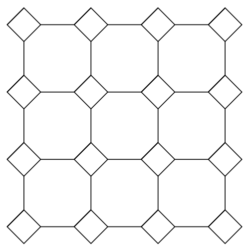


FIGURE 7.1

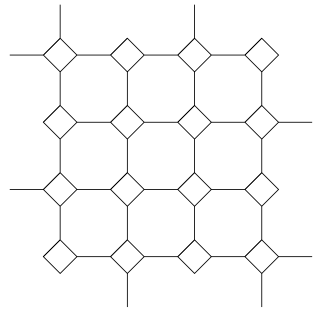


FIGURE 7.2(a)

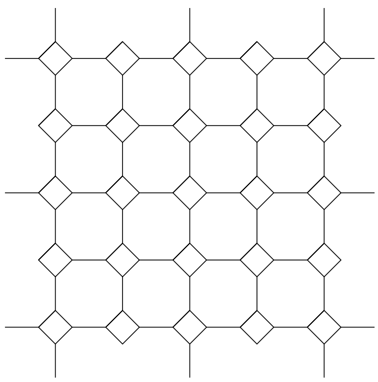


FIGURE 7.2(b)

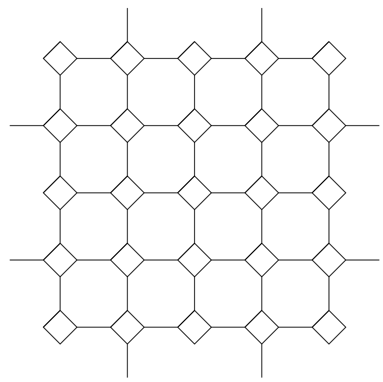


FIGURE 7.2(c)

These numbers are readily seen to satisfy the same recurrence as the $g_{n,k,h}$'s. Furthermore, it can be easily checked that $g_{3,k,h} = a_{3,k,h}$ for all k and h. Therefore $g_{n,k,h} = a_{n,k,h}$ for all n and we obtain from (6.1) the statement of the theorem. □

7. Fortresses

Consider the tiling of the plane by squares and regular octagons. Regard this tiling as a graph and let G_n be the subgraph induced by the vertices of n^2 4-cycles placed in an $n \times n$ array (G_4 is shown in Figure 7.1).

Add a leaf to every other vertex on each side of the boundary of G_n so that the two vertices in a corner end up having the same degree. For even n, this can be done in only one way up to isomorphism; let F_n denote the resulting graph. When n is odd, we obtain two essentially different graphs, which we denote by F_n and F'_n (by definition the former has leaves added to all corner vertices of G_n). The graphs obtained this way are called *fortress graphs* (or simply fortresses). The fortresses of order 4 and 5 are illustrated in Figure 7.2.

The statement of the following Theorem was first conjectured by Propp and later proved by Yang [**17**] using the permanent-determinant method (see e.g. [**10**] for a description of this method).

THEOREM 7.1 [YANG]. *For $n \geq 1$ we have*

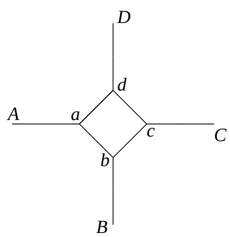


FIGURE 7.3(a)

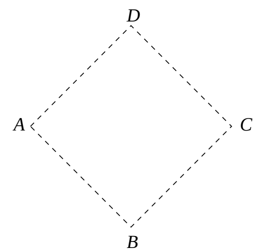


FIGURE 7.3(b)

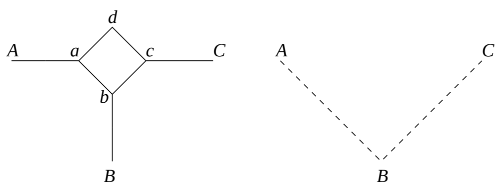


FIGURE 7.4(a)

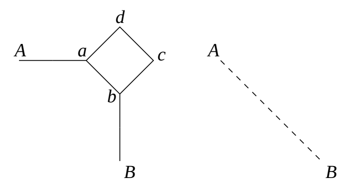


FIGURE 7.4(b)

$$M(F_{2n}) = 5^{n^2} \tag{7.1}$$

$$M(F_{4n+1}) = 5^{2n(2n+1)} \tag{7.2}$$

$$M(F_{4n-1}) = 2 \cdot 5^{2n(2n-1)} \tag{7.3}$$

$$M(F'_{4n+1}) = 2 \cdot 5^{2n(2n+1)} \tag{7.4}$$

$$M(F'_{4n-1}) = 5^{2n(2n-1)} \tag{7.5}$$

In this section we deduce this result as a consequence of the Complementation Theorem. This appears to be the first combinatorial proof of Yang's result.

In our proof we use the following observation due to Kuperberg [**9**], an instance of a principle called "urban renewal". Suppose a graph G has a subgraph isomorphic to the graph H shown in Figure 7.3(a) (all edges of H have weight 1). Suppose also that the vertices a, b, c and d have no neighbors outside H. Let $\bar{G}$ be the graph obtained from G by replacing H by the graph $\bar{H}$ shown in Figure 7.3(b) (dashed lines indicate edges of weight 1/2). Then an analysis of the restrictions of matchings of G and $\bar{G}$ to H and $\bar{H}$, respectively, shows that we have

$$M(G) = 2M(\bar{G}). \tag{7.6}$$

The above equality remains true when H and $\bar{H}$ are replaced by the pairs of graphs shown in Figures 7.4(a) and (b).

Proof of Theorem 7.1. Each matching of F_{2n} has $4n$ forced edges along the boundary. Apply the urban renewal trick described above $2n^2$ times to the graph obtained from F_{2n} after removing the forced edges (see Figure 7.5(a); when removing an edge we also remove

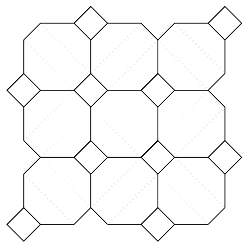


FIGURE 7.5(a)

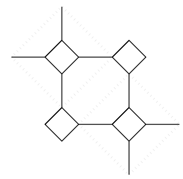


FIGURE 7.5(b)

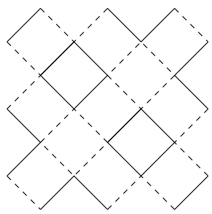


FIGURE 7.6

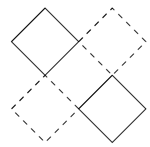


FIGURE 7.7

both its endpoints). The resulting graph is isomorphic to AD_{2n-1}, with half of its edges weighted by $1/2$ (see Figure 7.6). Therefore, (7.6) implies

$$M(F_{2n}) = 2^{2n^2} M(AD_{2n-1}; \mathrm{wt}), \tag{7.7}$$

where wt is the weight indicated in Figure 7.6. Since AD_{2n-1} and wt meet the conditions in the hypothesis of the Complementation Theorem, we obtain

$$M(AD_{2n-1}; \mathrm{wt}) = (1 + (1/2)^2)^{2n-1} M(AD_{2n-2}; \mathrm{wt}'), \tag{7.8}$$

where wt′ is the weight indicated in Figure 7.7. However, when performing urban renewal $(2n-2)^2/2$ times to F_{2n-2} *without* removing forced edges, one obtains an isomorphic weighting of AD_{2n-2} (see Figure 7.5(b)): it is obtained from the one in Figure 7.7 by rotation by 180 degrees. Therefore, (7.7), (7.8) and (7.6) imply

$$\begin{aligned} M(F_{2n}) &= 2^{(2n^2-(2n-2)^2/2)} \cdot (5/4)^{2n-1} M(F_{2n-2}) \\ &= 5^{2n-1} M(F_{2n-2}). \end{aligned}$$

Repeated application of this relation yields (7.1).

Similar arguments lead to the recurrences

$$M(F_{2n+1}) = 5^{2n} M(F'_{2n-1})$$

and

$$M(F'_{2n+1}) = 5^{2n} M(F_{2n-1}),$$

for all $n \geq 1$. Since $M(F_1) = 1$ and $M(F'_1) = 2$, we obtain (7.2)-(7.5). $\square$

Acknowledgments. I wish to thank John Stembridge for many useful discussions and for his continuing support during the writing of this paper. I would also like to thank James Propp for encouragement and for his interest in this work.

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