A VISUAL PROOF OF A RESULT OF KNUTH ON SPANNING TREES OF AZTEC DIAMONDS IN THE CASE OF ODD ORDER

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Abstract. The even Aztec diamond AD_n is known to have precisely four times more spanning trees than the odd Aztec diamond OD_n — this was conjectured by Stanley and first proved by Knuth. We present a short combinatorial proof of this fact in the case of odd n. Our proof works also for the more general case of odd by odd Aztec rectangles.

Suppose the corners of a $(2m+1)\times(2n+1)$ chessboard are black. The graph whose vertices are the unit squares of the board, and whose edges connect *diagonally* adjacent unit squares, has two connected components. The one whose vertices correspond to the white squares is denoted $AD_{m,n}$ and is called the *even Aztec rectangle* of order (m,n); the other is called the *odd Aztec rectangle* of order (m,n), and is denoted $OD_{m,n}$ (for $m=5$ and $n=3$ these are illustrated in Figure 1). For $m=n$ the even Aztec rectangle becomes the Aztec diamond graph, introduced in [4] and the subject of a considerable amount of research.

Stanley conjectured [7] that

$$\mathrm{t}(AD_{n,n}) = 4\,\mathrm{t}(OD_{n,n}) \tag{1}$$

for all $n\geq 1$, where $\mathrm{t}(G)$ denotes the number of spanning trees of the graph G. This was first proved by Knuth [5] by an algebraic method (finding explicitly the spectrum of the graphs). A vast generalization of this equality was given by Chow [1], also using an algebraic approach. We presented a weighted version of (1) in Theorem 5.1 of [3], with a combinatorial proof based on a certain complementation theorem for subgraphs of the grid graph (see Theorem 2.1 of [3]). The purpose of the present note is to present a short combinatorial proof of (1) in the case of odd n, as a direct consequence of the factorization theorem of [2] (see Theorem 1.2 there) and a construction of Temperley (see [8] or [6]) relating spanning trees and perfect matchings.

The form of the latter that we need can be stated as follows. Let G be an arbitrary graph obtained from the union of a finite number of 4-cycles of the grid graph $2\mathbb{Z}\times 2\mathbb{Z}$. Let G' be the graph obtained from G by splitting each of its 2×2 square faces into four 1×1 squares, and regarding the result as a subgraph of the grid $\mathbb{Z}\times\mathbb{Z}$ (Figure 2 illustrates the effect of a single splitting operation). Temperley's result (see [8] or [6]) states that for any vertex v of G on its infinite face, if $\mathrm{T}(G) := G'\setminus v$, then

$$\mathrm{t}(G) = \mathrm{M}(\mathrm{T}(G)), \tag{2}$$

where $\mathrm{M}(H)$ is the number of perfect matchings of the graph H. (Figure 3 shows two illustrations of this construction. The first corresponds to $AD_{5,3}$. The second was obtained from $OD_{5,3}$ by first removing the four vertices of degree 1, then splitting each face, and finally removing an appropriate vertex from the infinite face).

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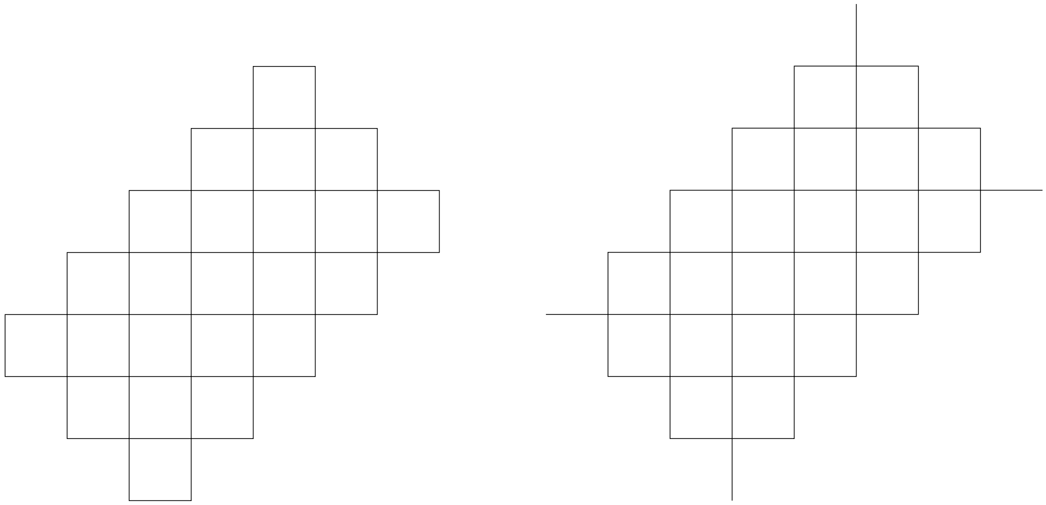

FIGURE 1. (a) The even Aztec rectangle $AD_{5,3}$. (b) The odd Aztec rectangle $OD_{5,3}$.

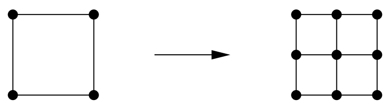

FIGURE 2. The operation of splitting a square face into four smaller square faces.

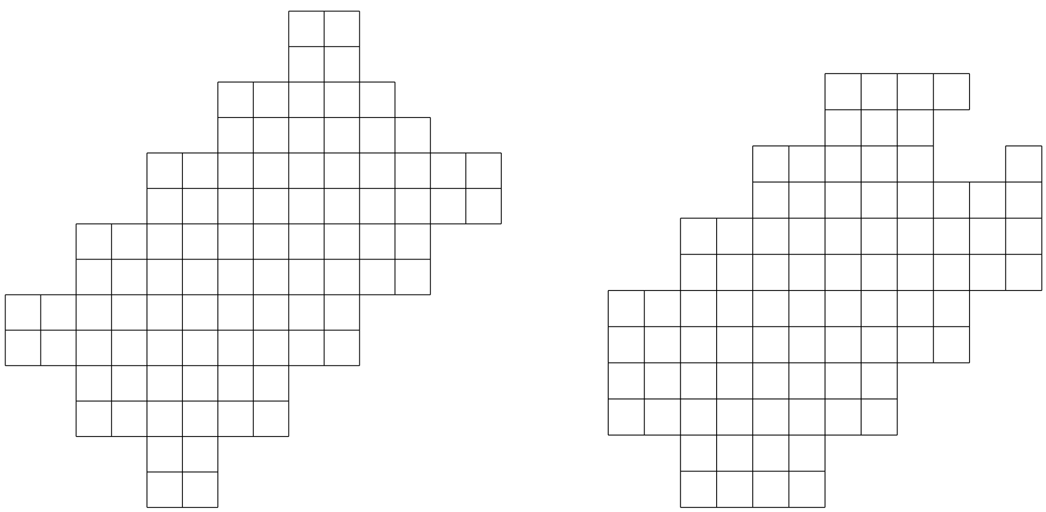

FIGURE 3. Graphs corresponding by Temperley's construction to: (a) $AD_{5,3}$; (b) the graph obtained from $OD_{5,3}$ by removing its four vertices of degree 1.

Let G be a connected subgraph of the grid $\mathbb{Z}^2$ symmetric about a diagonal lattice line ℓ. Assume all the vertices of G on ℓ are consecutive lattice points on ℓ. Scan these vertices from left to right, and alternate between deleting the edges of G that touch them from above, and deleting the edges of G that touch them from below (Figure 4 illustrates the result of these operations for the graphs in Figure 3); let G^+ and G^- be the connected components of the resulting graph that are above and below ℓ, respectively. It is easy to see that the number of vertices of G on ℓ must be even if G admits perfect matchings; let $\mathrm{w}(G)$ be half this number.

Then the Factorization Theorem of [2] implies that

$$\mathrm{M}(G) = 2^{\mathrm{w}(G)}\,\mathrm{M}(G^+)\,\mathrm{M}(G^-). \tag{3}$$

We are now ready to show how for odd n (1) follows as a direct consequence of (2) and (3).

Theorem 1. *For all odd integers $m, n \geq 1$ we have $\mathrm{t}(AD_{m,n}) = 4\,\mathrm{t}(OD_{m,n})$.*

Proof. Let $\mathrm{T}(AD_{m,n})$ be the graph obtained by Temperley's construction from $AD_{m,n}$ by choosing v to be the rightmost vertex of $AD_{m,n}$ on its southwest-northeast going symmetry axis (for $m = 5$ and $n = 3$ this is shown in Figure 3(a)). Applying (3) to it we obtain

$$\mathrm{M}(\mathrm{T}(AD_{m,n})) = 2^m\,\mathrm{M}(G_1^+)\,\mathrm{M}(G_1^-), \tag{4}$$

where G_1^+ and G_1^- are obtained from $\mathrm{T}(AD_{m,n})$ as described in the paragraph before (3); for $m = 5$, $n = 3$, they are illustrated in Figure 4(a).

Before we handle the odd diamond $OD_{m,n}$ similarly, it will be convenient to change slightly its definition, namely by removing its four leaves. This clearly leaves the number of its spanning trees—and thus the statement of the Theorem—unchanged.

Let $\mathrm{T}(OD_{m,n})$ be the graph obtained by Temperley's construction from this leafless odd diamond by choosing v to be its rightmost vertex of on its southwest-northeast going symmetry axis (for $m = 5$ and $n = 3$ this is pictured in Figure 3(b)). Applying (3) to it we obtain

$$\mathrm{M}(\mathrm{T}(OD_{m,n})) = 2^{m-1}\,\mathrm{M}(G_2^+)\,\mathrm{M}(G_2^-), \tag{5}$$

where G_2^+ and G_2^- are obtained analogously from $\mathrm{T}(OD_{m,n})$; for $m = 5$, $n = 3$, they are illustrated in Figure 4(b).

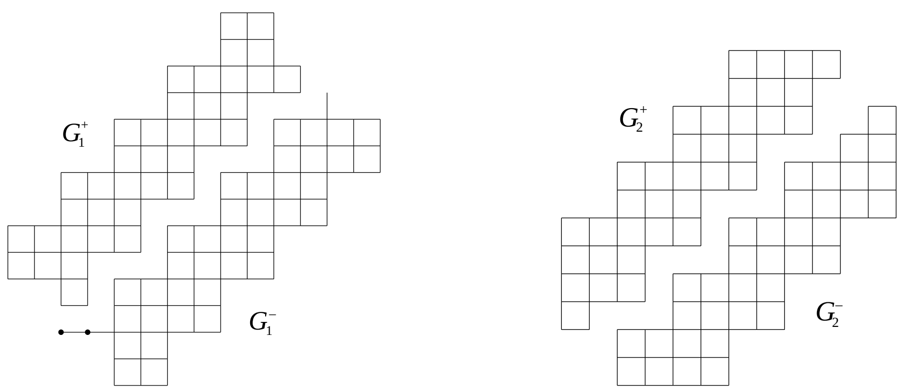

FIGURE 4. The effect of the Factorization Theorem on the graphs corresponding by Temperley's construction to: (a) $AD_{5,3}$; (b) $OD_{5,3}$.

Note that G_1^- has two vertices of degree 1, and the edges incident to them must be present in all its perfect matchings. However, the graph obtained from G_1^- by removing the vertices matched by these two forced edges on the one hand, and G_2^- on the other hand, are readily seen to be the results of Temperley's construction applied to isomorphic graphs, with different choices for the removed vertex v (see Figure 4). Thus, (4) and (5) imply

$$\frac{\mathrm{M}(\mathrm{T}(AD_{m,n}))}{\mathrm{M}(\mathrm{T}(OD_{m,n}))} = 2\frac{\mathrm{M}(G_1^+)}{\mathrm{M}(G_2^+)}. \tag{6}$$

Furthermore, G_1^+ and G_2^+ both admit symmetry axes that are diagonal lattice lines (going northwest-southeast). Apply (3) to each of them (for $m = 5$, $n = 3$, this is illustrated in Figure 5).

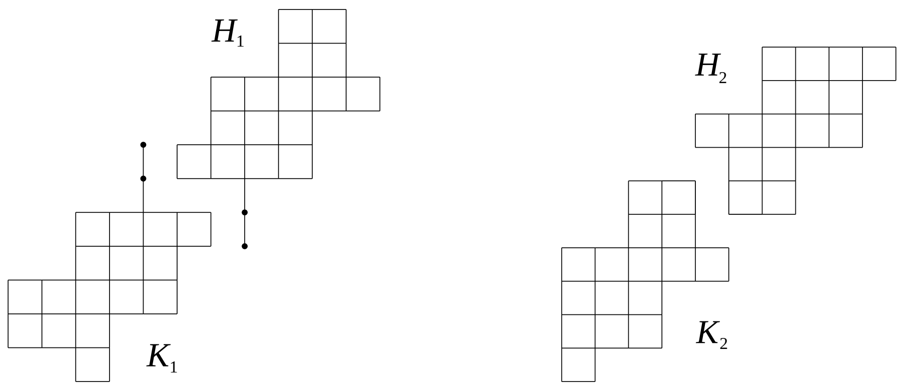

FIGURE 5. The effect of the Factorization Theorem on: (a) the top part of Figure 4(a); (b) the top part of Figure 4(a).

We get:

$$\mathrm{M}(G_1^+) = 2^{(n+1)/2}\,\mathrm{M}(H_1)\,\mathrm{M}(K_1) \tag{7}$$

$$\mathrm{M}(G_2^+) = 2^{(n-1)/2}\,\mathrm{M}(H_2)\,\mathrm{M}(K_2), \tag{8}$$

where H_1 (resp., K_1) and H_2 (resp., K_2) are the resulting subgraphs above (resp., below) the symmetry axes in G_1^+ and G_2^+, respectively. However, one readily sees that the graph obtained from H_1 after removing its one forced edge is isomorphic to H_2 (being its rotation by 180°), and the graph obtained from K_1 after removing its forced edge is isomorphic to K_2 (as it is obtained by reflecting across the horizontal the 90° rotation of K_2). Thus (7) and (8) imply $M(G_1^+) = 2M(G_2^+)$, and hence by (6) we have $\mathrm{M}(\mathrm{T}(AD_{m,n})) = 4\,\mathrm{M}(\mathrm{T}(OD_{m,n}))$. The statement of the Theorem follows now by (2). □

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