HIGHER DIMENSIONAL AZTEC DIAMONDS AND A (2^d+2)-VERTEX MODEL

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ABSTRACT. Motivated by the close relationship between the number of perfect matchings of the Aztec diamond graph introduced in [5] and the free energy of the square-ice model, we consider a higher dimensional analog of this phenomenon. For $d \geq 1$, we construct d-uniform hypergraphs which generalize the Aztec diamonds and we consider a companion d-dimensional statistical model (called the 2^d+2-vertex model) whose free energy is given by the logarithm of the number of perfect matchings of our hypergraphs. We prove that the limit defining the free energy per site of the 2^d+2-vertex model exists and we obtain bounds for it. As a consequence, we obtain an especially good asymptotical approximation for the number of matchings of our hypergraphs.

1. Introduction

In [**5**] there are introduced the Aztec diamond graphs, which can be defined as follows. Consider a $(2n+1)\times(2n+1)$ chessboard with black corners. The graph whose vertices are the white squares and whose edges connect precisely those pairs of white squares that are diagonally adjacent is called the *Aztec diamond* of order n, and is denoted AD_n (Figure 1.1 illustrates the case $n=5$).

A *perfect matching* of a graph is a collection of vertex-disjoint edges collectively incident to all vertices. We will often refer to a perfect matching simply as a *matching*. The number of matchings of a graph G is denoted by $M(G)$.

The number of perfect matchings of AD_n is given by the simple formula $M(AD_n) = 2^{n(n+1)/2}$ (see [**5**] and [**2**]).

The work in this paper was motivated by the following. Modify the definition of the Aztec diamond by replacing the $(2n+1)\times(2n+1)$ chessboard by a $2n\times 2n$ *toroidal* chessboard. The resulting graph, denoted TD_n, is called the *toroidal Aztec diamond* of order n. What can be said about $M(TD_n)$?

As noted for example in [**5**], this question is closely related to the *square-ice model* of statistical mechanics, solved by Lieb [**6**], [**7**], [**8**] and Sutherland [**11**]. More precisely, the limit $\lim_{n\to\infty}(1/n^2)\log M(TD_n)$ turns out to be the free energy per site of this model, for a particular choice of Boltzmann weights.

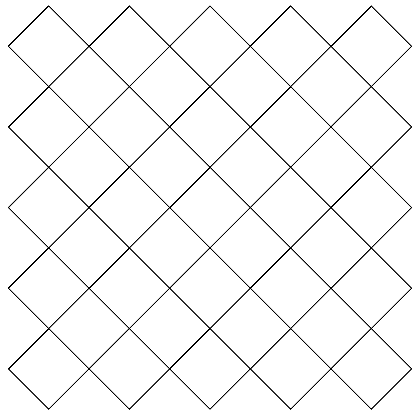

FIGURE 1.1

Guided by an alternative description of the matchings of the toroidal Aztec diamond, in Section 2 we construct hypergraphs, denoted $TD_{n_1,\cdots,n_d}$, which may be regarded as d-dimensional generalizations of TD_n. The limit

$$L_d = \lim_{n\to\infty}(1/n^d)\log M(TD_{n^{(d)}}) \tag{1.1}$$

(where $n^{(d)}$ stands for d subscripts equal to n and $M(H)$ denotes the number of perfect matchings of the hypergraph H) turns out to be the free energy of a certain d-dimensional generalization of the square-ice model, for a suitable choice of Boltzmann weights.

The model arising this way is a vertex model on the $\mathbf{Z}^d$ lattice, with two types of admissible arrow configuration around a vertex v: *balanced* configurations, in which each pair of collinear edges incident to v are oriented in the same direction; and *special* configurations, in which either all edges point towards v or all point away from v. We weight the special and balanced states by a and b, respectively ($a, b \geq 0$). Since the partition function is a homogeneous function of the weights, we may assume without loss of generality that $b = 1$.

For the sake of notational simplicity, in the indexing set of an object $\mathcal{O}$ we will often denote by $n^{(d)}$ a sequence of d integers equal to n. Moreover, we will let $\mathcal{O}_n^{(d)}$ stand for $\mathcal{O}_{n^{(d)}}$ (for example, we write $TD_n^{(d)}$ for the hypergraph on the right hand side of (1.1)).

We employ the transfer matrix method (see e.g. [**10**]) to prove that the limit defining the free energy per site of our d-dimensional model exists (see Theorem 3.5). The main result of this paper is Theorem 3.10, which gives bounds for the free energy per site. As a corollary, we obtain that $L_d = ((d-1)/2)\log 2 + \varepsilon_d$, where $0 < \varepsilon_d < 2^{-d-(d-3)2^{d-2}}$.

2. Higher dimensional toroidal Aztec diamonds and the $(2^d + 2)$-vertex model

For the purpose of constructing higher dimensional analogs of the Aztec diamond we will find it convenient to view the matchings of AD_n as follows. Let G_n be the subgraph of the grid $\mathbf{Z}^2$ induced by the vertices having non-negative coordinates not exceeding n. The union of any two incident edges of G_n that form a $90°$ angle is called a *2-claw*. A partition of the edges of G_n into 2-claws is called a 2-claw covering. Clearly, there is a bijection between 2-claw coverings of G_n and perfect matchings of the graph whose vertices are the

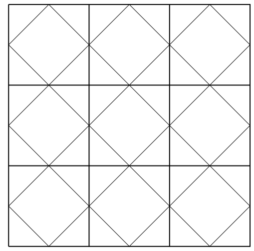

FIGURE 2.1

midpoints of edges of G_n, with an edge connecting the midpoints of e and f precisely if $e \cup f$ is a 2-claw. However, this graph is isomorphic to AD_n (see Figure 2.1).

Therefore, the matchings of AD_n can be identified with 2-claw coverings of G_n. This point of view is useful because it provides the setting for the following very natural generalization. Let $G_n^{(d)}$ be the subgraph of the d-dimensional grid graph $\mathbf{Z}^d$ induced by the vertices having non-negative coordinates not exceeding n. Define a *d-claw* to be the union of any d pairwisely orthogonal edges of $G_n^{(d)}$ that are incident to a common vertex. The question then is to determine the number of d-claw coverings of $G_n^{(d)}$.

Just as for $d = 2$, we can rephrase this as a matching problem as follows. Let $AD_n^{(d)}$ be the uniform d-hypergraph whose vertices are the midpoints of edges of $G_n^{(d)}$, with d vertices connected by an edge precisely if they are midpoints of edges of $G_n^{(d)}$ that form a d-claw. Then the d-claw coverings of $G_n^{(d)}$ can be identified with perfect matchings of $AD_n^{(d)}$ (by a perfect matching of a hypergraph we mean a collection of vertex-disjoint edges that are collectively incident to all vertices).

The edges of $AD_n^{(d)}$ can be visualized as follows. Consider an interior vertex v of $G_n^{(d)}$ (i.e., no coordinate of v is 0 or n). The midpoints of the $2d$ edges incident to v form a regular (d-dimensional) octahedron O_d centered at v. Consider translations of O_d centered at each vertex of $G_n^{(d)}$ (disregard the vertices of these translates whose coordinates do not all fall in the range $[0, n]$). Then the edges of $AD_n^{(d)}$ are precisely the $(d-1)$-dimensional faces of these octahedra.

In the study of statistical-mechanical models it is customary to consider toroidal boundary conditions, i.e., to identify corresponding vertices on opposite faces of the "crystal". Let $T_n^{(d)}$ and $TD_n^{(d)}$ be the graphs obtained by applying this procedure to $G_n^{(d)}$ and $AD_n^{(d)}$, respectively (in particular, $TD_n^{(2)}$ is the graph TD_n defined at the beginning of the Introduction). Then the edges of $TD_n^{(d)}$ are precisely the $(d-1)$-faces of n^d octahedra that touch only at vertices, which we will refer to as *cells*.

The use of the same name as in the case of the "cellular graphs" of [**3**] is not accidental (a graph is said to be *cellular* if its edges can be partitioned into 4-cycles so that at most two of them meet at a vertex). Indeed, all the results in Section 2 of [**3**] have correspondents for *cellular d-hypergraphs*, i.e., uniform d-hypergraphs whose edges can be partitioned into cells so that each vertex is contained in at most two cells (see also Section 5 of [**2**]).

To illustrate this, consider for example $TD_n^{(d)}$. Its cells can be naturally grouped in "lines," so that each cell is contained in precisely d lines (collect in a line the cells whose centers have all but one of their coordinates identical). For the case under consideration each line L is in fact a "cycle," i.e., every cell of L is bordered by two other cells of L.

Let μ be a perfect matching of $TD_n^{(d)}$. Assign one of the numbers 1, 0 or -1 to each cell of $TD_n^{(d)}$ according as the cell contains 2, 1 or 0 edges of μ (two is the maximum number of disjoint $(d-1)$-faces in an octahedron of dimension d). Then by an argument similar to the one used to prove Lemma 2.2 of [**3**] it can be shown that the obtained pattern A is "sign-alternating" (i.e., the nonzero elements alternate in sign along each cycle).

Conversely, consider an assignment of 1's, 0's and -1's to the cells of $TD_n^{(d)}$ forming an alternating sign pattern (for short, ASP) A. Let $\mathcal{C}$ be the collection of cycles of A consisting entirely of zeroes. Using the argument in the proof of Lemma 2.3 of [**3**], we obtain that, once we fix orientations on the cycles in $\mathcal{C}$, the matchings compatible with these orientations and having corresponding pattern A are uniquely determined on the 0-cells and can be freely chosen (from 2^{d-1} possibilities) on the 1-cells.

Consider now in the same picture the graph $T_n^{(d)}$, with vertices at the centers of the cells of $TD_n^{(d)}$ and edges passing through these cells. If v is the center of a 1-cell (resp., -1-cell) then orient all edges of $T_n^{(d)}$ incident to v so that they point away from (resp., towards) v; call such arrow configurations around a vertex *special.* Finally, if v is the center of a 0-cell c, orient the edges of $T_n^{(d)}$ along each cycle containing c so that they point away from the 1-cell and towards the -1-cell bordering the run of zeroes containing c (in case c is contained in a cycle of zeroes, pick one of the two orientations of the corresponding cycle of $T_n^{(d)}$ with all edges pointing in the same direction along the cycle). This results in an arrow configuration around v such that from the two edges incident to v parallel to any coordinate axis one points towards v and the other away from v; we call such vertex configurations *balanced* (the four balanced and the two special vertex configurations corresponding to $d=2$ are shown in Figure 2.2(a)). Call an orientation of $T_n^{(d)}$ *admissible* if the arrow configuration around each vertex is either balanced or special. Then the preceding paragraph shows that there is a natural correspondence between perfect matchings of $TD_n^{(d)}$ and admissible orientations of $T_n^{(d)}$, with precisely $2^{(d-1)N}$ matchings corresponding to an admissible orientation having N special vertex configurations pointing outward.

This suggests considering the following model, which we will call the (2^d+2)*-vertex model.* Consider the set of admissible orientations of $T_n^{(d)}$. Weight the balanced vertex configurations by 1 and the special ones by $a \geq 0$. The weight of an admissible orientation is the product of weights of all vertex configurations. The *partition function,* denoted $Z_n^{(d)}(a)$, is the sum of the weights of all admissible orientations.

Let C be a cycle of $T_n^{(d)}$ consisting of n vertices with $d-1$ of their coordinates identical and consider an admissible orientation of $T_n^{(d)}$. Then along C, the two special configurations occur alternately. Therefore, the two special configurations appear the same number of times in every admissible orientation. This shows that the partition function is not affected by changing the weight of one special configuration to a^2 and the other to 1. The above arguments show then that

$$M(TD_n^{(d)}) = Z_n^{(d)}(2^{(d-1)/2}). \tag{2.1}$$

For $d=2$, the above described model is equivalent to the square-ice (also known as six-vertex) model. Indeed, reverse arrows on all horizontal segments in each admissible

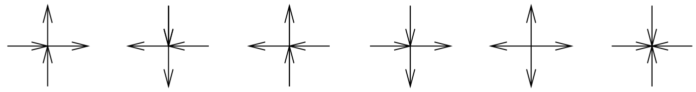

FIGURE 2.2(a)

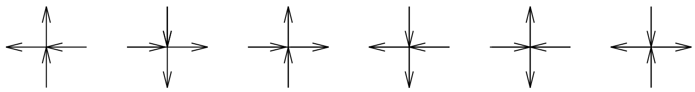

FIGURE 2.2(b)

orientation of $T_n^{(2)}$. This amounts to changing the admissible configurations around a vertex to the ones showed in Figure 2.2(b). However, these are precisely the allowed local arrangements in the square-ice model (see e.g. [**1**, p.128]).

An important characteristic of a statistical model is the *free energy per site*, defined (up to a multiplicative constant) to be the limit $\lim_{N\to\infty}(1/N)\log Z$, where Z is the partition function and N is the number of "particles" in the system. This limit is expected to exist, from a physical point of view. We prove (Theorem 3.5) that this limit does indeed exist for our d-dimensional model. Our method can in fact be applied to prove the existence of this limit for a large class of statistical models (see Remark 3.6).

3. Bounds for the free energy of the (2^d+2)-vertex model

Let $G_{n_1,\dots,n_d}$ be the subgraph of the d-dimensional grid $\mathbf{Z}^d$ induced by the vertices with coordinates $(x_1,\dots,x_d)$ satisfying $0\le x_i\le n_i$, $i=1,\dots,d$. We will find it useful to enlarge the set of objects under consideration to admissible orientations of the toroidal grid $T_{n_1,\dots,n_d}$ obtained by identifying vertices of $G_{n_1,\dots,n_d}$ having equal i-th coordinates modulo n_i, $i=1,\dots,d$.

Our derivation of an upper bound for $Z_n^{(d)}(a)$ relies on the following ideas. First, we use the transfer matrix method to encode the admissible orientations of our toroidal grid $T_{n_1,\dots,n_d}$ as closed walks in a certain weighted directed graph S. More precisely, $Z_n^{(d)}(a)$ turns out to be the trace of the n_d-th power of the adjacency matrix A of S, so it equals the sum of the n_d-th powers of its eigenvalues. A simple but crucial observation is that the matrix A is symmetric, and therefore its eigenvalues are real. The other key observation is that our problem is invariant under permutations of the d coordinates, so in particular, for any choice of $i\in\{1,\dots,d\}$, we could have constructed S such that $Z_n^{(d)}(a)$ is the trace of the n_i-th power of its adjacency matrix. These two simple facts allow us to prove Lemma 3.2, and then deduce a family of upper bounds for the free energy per site (see inequality (3.5); an application of bounds analogous to these for the case of the three dimensional dimer problem is given in [**4**]).

However, in order for the bounds (3.5) to be effective, one needs to be able to determine (or estimate) the largest eigenvalue of the matrix A (whose entries involve the parameter a) for some particular (even) values of the n_i's, and this seems to be very difficult even for small values. The way we resolve this is by proving Lemma 3.8, which relates the partition function in dimension d to the one in dimension $d-1$. The statement of Lemma 3.8 was suggested by Lemma 3.2 and the fact that, when one of the n_i's is 2, all the information in the admissible d-dimensional configurations is contained in the admissible orientations of a grid in one dimension lower. Proposition 3.9 and Theorem 3.10 are then deduced easily.

Let S be the graph consisting of the $n_1 n_2 \cdots n_{d-1}$ disjoint edges

$$[(x_1, \ldots, x_{d-1}, 0), (x_1, \ldots, x_{d-1}, 1)],$$

$0 \leq x_i < n_i$, $i = 1, \ldots, d-1$.

Define a weighted directed graph D as follows. Take the vertices of D to be the $2^{n_1 \cdots n_{d-1}}$ orientations of S. Let α and β be two such orientations. Translate α such that its edges are the segments $[(x_1, \ldots, x_{d-1}, -1), (x_1, \ldots, x_{d-1}, 0)]$, $0 \leq x_i < n_i$, $i = 1, \ldots, d-1$. Let G be the subgraph of $T_{n_1, \ldots, n_d}$ contained in the hyperplane $x_d = 0$ (G is isomorphic to $T_{n_1, \ldots, n_{d-1}}$). Define the weight of the edge from α to β to be the total weight of the orientations of G which give rise only to admissible (d-dimensional) arrow configurations around the vertices of G (if no such orientation exists the weight is taken to be zero).

We claim that the total weight of the admissible orientations of $T_{n_1, \ldots, n_d}$ (i.e., the partition function $Z_{n_1, \ldots, n_d}$) is equal to the total weight of the closed walks of length n_d in D (if $e_1 e_2 \cdots e_n$ is a closed walk, then $e_i e_{i+1} \cdots e_n e_1 \cdots e_{i-1}$ is in general a different closed walk).

Indeed, by considering the hyperplanes $H_i : x_d = i$, $i = 0, \ldots, n_d - 1$, any admissible orientation of $T_{n_1, \ldots, n_d}$ can be regarded as a closed walk of length n_d in the above constructed graph D. It is clear that the weight of any such given walk C is equal to the total weight of admissible orientations of $T_{n_1, \ldots, n_d}$ with configurations between the hyperplanes H_i specified by the vertices of C.

Pick a linear order on the vertices of D and let A be the transfer matrix, i.e. the matrix whose (i, j) entry is equal to the weight of the edge from i to j. Let $\lambda_{n_1, \ldots, n_{d-1}; l}$ be the eigenvalues of A ($l = 1, \ldots, 2^{n_1 \cdots n_{d-1}}$). Then by the Transfer Matrix Theorem [**10**, Corollary 4.7.3], we obtain

$$Z_{n_1, \ldots, n_d} = \sum_l \left(\lambda_{n_1, \ldots, n_{d-1}; l} \right)^{n_d}. \tag{3.1}$$

LEMMA 3.1. *The matrix A is symmetric.*

Proof. Let i and j be two vertices of D and let G be, as before, the subgraph of $T_{n_1, \ldots, n_d}$ induced by the vertices with d-th coordinate zero. Define $\mathcal{M}_{ij}$ (resp., $\mathcal{M}_{ji}$) to be the set of admissible orientations of G compatible with the transition from vertex i to vertex j (resp., vertex j to vertex i).

Given $\alpha \in \mathcal{M}_{ij}$, let α' be the orientation of G obtained by reversing all arrows in α. We claim that $\alpha' \in \mathcal{M}_{ji}$.

Indeed, let v be a vertex of G. Suppose the oriented segments in i and j are positioned as in the definition of the weight from i to j. Then v is incident to $2d-2$ edges of G (which we leave unoriented for the moment) and to one oriented edge in both i and j. Write (i, j) to express the fact that i is followed by j. The only instances when the neighborhood of v is not the same for (i, j) as for (j, i) occur when the corresponding edges of i and j have opposite orientations: in such cases the edges point towards v for (i, j) if and only if they point away from v for (j, i).

However, in this case the orientation of G in the neighborhood of v is forced to be the appropriate special vertex configuration, and all we have to do to pass from the orientation determined by (i, j) to the one determined by (j, i) is reverse all arrows in G. Since the operation of reversing all arrows in G preserves balanced arrow configurations around its vertices, we obtain our claim.

By interchanging the roles of i and j, we obtain that there is a similar map from $\mathcal{M}_{ji}$ to $\mathcal{M}_{ij}$, and it follows from our construction that the two maps are inverse to one another. Therefore, the map $\alpha \mapsto \alpha'$, which is clearly weight-preserving, is a bijection. This implies that $A_{ij} = A_{ji}$. $\square$

Since the entries of A are nonnegative, the Perron-Frobenius theorem (see e.g. [**9**]) implies that A has an eigenvalue $\lambda_{n_1,\dots,n_{d-1}} \geq 0$ greater or equal than the absolute value of all remaining eigenvalues.

LEMMA 3.2. *Let $n, k \geq 2$ be even. Then for all nonnegative $i \leq d-2$ we have*

$$\frac{1}{k^i n^{d-i-1}} \log \lambda_{k^{(i)} n^{(d-i-1)}} \leq \frac{1}{n} \log 2 + \frac{1}{k^{i+1} n^{d-i-2}} \log \lambda_{k^{(i+1)} n^{(d-i-2)}}$$

(recall that $m^{(j)}$ denotes a sequence of j subscripts equal to m).

Proof. By Lemma 3.1, the eigenvalues of A are real. Since k is even, we have by (3.1)

$$(\lambda_{k^{(i)} n^{(d-i-1)}})^k \leq \sum_j (\lambda_{k^{(i)} n^{(d-i-1)};j})^k = Z_{k^{(i)} n^{(d-i-1)},k}. \tag{3.2}$$

However, since our model is invariant under permutation of coordinates, we have that $Z_{k^{(i)} n^{(d-i-1)},k}$ equals $Z_{k^{(i+1)} n^{(d-i-1)}}$. Using (3.1) and the fact that n is even we obtain

$$Z_{k^{(i+1)} n^{(d-i-1)}} = \sum_j (\lambda_{k^{(i+1)} n^{(d-i-2)};j})^n \leq 2^{k^{i+1} n^{d-i-2}} (\lambda_{k^{(i+1)} n^{(d-i-2)}})^n. \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$(\lambda_{k^{(i)} n^{(d-i-1)}})^k \leq 2^{k^{i+1} n^{d-i-2}} (\lambda_{k^{(i+1)} n^{(d-i-2)}})^n.$$

Taking the logarithm of both sides and dividing by $k^{i+1} n^{d-i-1}$ we obtain the statement of the Lemma. $\square$

COROLLARY 3.3. *For n, k even we have*

$$\frac{1}{n^{d-1}} \log \lambda_n^{(d-1)} \leq \frac{d-1}{n} \log 2 + \frac{1}{k^{d-1}} \log \lambda_k^{(d-1)}. \tag{3.4}$$

Proof. Apply Lemma 3.2 for $i = 0, 1, \dots, d-2$. We obtain a chain of inequalities that implies the statement of the Corollary. $\square$

LEMMA 3.4. *The sequence $\{(1/(2n)^{d-1}) \log \lambda_{2n}^{(d-1)}\}_n$ is convergent.*

Proof. Let $\overline{l}$ and $\underline{l}$ be the superior and inferior limit of the sequence in the statement of the Lemma, respectively. By taking the superior limit as $n \to \infty$ of both sides of (3.4) we obtain

$$\overline{l} \leq \frac{1}{(2k)^{d-1}} \log \lambda_{2k}^{(d-1)}, \tag{3.5}$$

for all $k \geq 1$. However, taking the inferior limit of both terms of the above inequality as $k \to \infty$ we obtain $\overline{l} \leq \underline{l}$, which completes the proof. □

Denote by $l_d(a)$ the limit of the sequence in the statement of Lemma 3.4, where a is the weight of the two special vertex configurations. Recall that $Z_n^{(d)}(a)$ is the partition function when the special configurations are weighted by a.

THEOREM 3.5.

$$\lim_{n\to\infty} \frac{1}{n^d} \log Z_n^{(d)}(a) = l_d(a). \tag{3.6}$$

Proof. We show first that for $n \geq 1$ and $i = 1, \ldots, d$ we have

$$Z_{n^{(i)}(n+1)^{(d-i)}}(a) \leq Z_{n^{(i-1)}(n+1)^{(d-i+1)}}(a). \tag{3.7}$$

Indeed, we can regard the two partition functions as being the generating functions for closed walks of length n and respectively $n+1$ in a suitable directed graph D. However, each closed walk C of length n can be augmented to a closed walk C' of length $n+1$ by inserting a loop at a vertex, since D has loops at all vertices. Therefore, as the weight of C' is clearly at least as large as the weight of C, we obtain (3.7).

Repeated application of (3.7) implies that $Z_n^{(d)}(a) \leq Z_{n+1}^{(d)}(a)$, for all $n \geq 1$. In view of this, to prove (3.6) it suffices to show that the even-index terms of the sequence on the left hand side converge to $l_d(a)$.

Let therefore n be even. Using (3.1) and the fact that the eigenvalues are real we obtain

$$(\lambda_n^{(d-1)})^n \leq Z_n^{(d)}(a) = \sum_j (\lambda_{n^{(d-1)};j})^n \leq 2^{n^{d-1}} (\lambda_n^{(d-1)})^n.$$

Taking the logarithm and dividing by n^d we are led to

$$\frac{1}{n^{d-1}} \log \lambda_n^{(d-1)} \leq \frac{1}{n^d} \log Z_n^{(d)}(a) \leq \frac{1}{n} \log 2 + \frac{1}{n^{d-1}} \log \lambda_n^{(d-1)}.$$

Using Lemma 3.4 and letting $n \to \infty$ we obtain (3.6). □

REMARK 3.6. The argument in the above proof can be used to prove the existence of the corresponding limit for any (d-dimensional) statistical model provided

(i) the set of admissible configurations is invariant under permutations of the coordinates

(ii) the admissible configurations can be interpreted as closed walks in a weighted directed graph D

(iii) the incidence matrix of D is symmetric and all diagonal entries are positive.

In particular, using a suitable modification of (iii), we can apply this argument to the dimer problem on the d-dimensional toroidal grid $T_{2n}^{(d)}$.

LEMMA 3.7. $Z_n(a) = (1+a)^n + (1-a)^n$.

Proof. The graph T_n is a cycle; all its orientations are admissible and contain the same number of each type of special configurations. The total weight of the orientations containing exactly $2i$ special configurations is $2\binom{n}{2i}a^{2i}$. We obtain

$$Z_n(a) = 2\sum_i \binom{n}{2i} a^{2i} = (1+a)^n + (1-a)^n.$$

□

The subgraph of $T_{n_1,\ldots,n_d}$ induced by the vertices that have all but one of their coordinates identical is called a *circuit*. In case an orientation is given, a circuit is *monotone* if all edges point in the same direction along the circuit.

Given an admissible orientation of $T_{n_1,\ldots,n_d}$, assign value zero to its balanced vertices and values 1 and -1 to the special vertices with arrows pointing outward and inward, respectively. The resulting array, which is said to have shape $(n_1,\ldots,n_d)$, is clearly an alternating sign pattern (i.e., the nonzero entries alternate in sign along each circuit). Let $ASP(n_1,\ldots,n_d)$ be the set of alternating sign patterns of shape $(n_1,\ldots,n_d)$. We claim that given such a pattern A, there are precisely $2^{z(A)}$ admissible orientations giving rise to A, where $z(A)$ is the number of cycles of A consisting entirely of zeroes (a cycle of A is the set of entries along a circuit).

Indeed, A determines the orientation along all circuits corresponding to cycles of A containing nonzero elements. However, circuits corresponding to the remaining cycles of A must be monotone and can therefore be oriented in two different ways.

The following result is crucial in obtaining an upper bound for $l_d(a)$.

LEMMA 3.8.

$$Z_{2,n^{(d-1)}}(a) \le 2^{n^{d-1}+(d-1)n^{d-2}} Z_n^{(d-1)}(a^2/2).$$

Proof. An alternating sign pattern of shape $(2, n^{(d-1)})$ can be regarded as a pair (A, B), $A, B \in ASP(n^{(d-1)})$, where A and B are so that their corresponding entries form 2-cycles along which the nonzero elements alternate in sign. Since this implies $A = -B$, we may in fact identify $ASP(2, n^{(d-1)})$ with $ASP(n^{(d-1)})$. Using the correspondence between admissible orientations and ASP's mentioned above we obtain that

$$Z_{2,n^{(d-1)}}(a) = \sum_{A\in ASP(n^{(d-1)})} 2^{z(A)} a^{N_+(A)+N_-(A)} \cdot 2^{z(A)} a^{N_-(A)+N_+(A)} \cdot 2^{N_0(A)}, \qquad (3.8)$$

where $N_0(A)$ is the number of zeroes in A.

Since $z(A)$ cannot exceed $(d-1)n^{d-2}$, the total number of cycles of A, and since $N_0(A) = n^{d-1} - N_+(A) - N_-(A)$, we deduce from (3.8) that

$$Z_{2,n^{(d-1)}}(a) \le 2^{n^{d-1}+(d-1)n^{d-2}} \sum_{A\in ASP(n^{(d-1)})} 2^{z(A)} (a^2/2)^{N_+(A)+N_-(A)}.$$

Using again the correspondence between admissible orientations and ASP's we identify the sum on the right hand side as being $Z_n^{(d-1)}(a^2/2)$, thus completing the proof. □

PROPOSITION 3.9. $l_d(a) \le (1/2)\log 2 + (1/2) l_{d-1}(a^2/2)$.

Proof. Let n be even. By Lemma 3.2 we have that

$$\frac{1}{n^{d-1}} \log \lambda_n^{(d-1)} \le \frac{1}{n} \log 2 + \frac{1}{2n^{d-2}} \log \lambda_{2,n^{(d-2)}}.$$

Take the superior limit of both sides as n approaches infinity by even values. By Theorem 3.5, this gives rise on the left hand side to $l_d(a)$ and we obtain

$$\begin{aligned} l_d(a) &\le \limsup_{n\to\infty, n \text{ even}} \frac{1}{2n^{d-2}} \log \lambda_{2,n^{(d-2)}} \\ &\le \limsup_{n\to\infty, n \text{ even}} \frac{1}{2n^{d-1}} \log \sum_j (\lambda_{2,n^{(d-2)};j})^n \\ &= \limsup_{n\to\infty, n \text{ even}} \frac{1}{2n^{d-1}} \log Z_{2,n^{(d-1)}}(a). \end{aligned} \tag{3.9}$$

By Lemma 3.8 we have

$$\log Z_{2,n^{(d-1)}}(a) \le (n^{d-1} + (d-1)n^{d-2}) \log 2 + \log Z_n^{(d-1)}(a^2/2).$$

Dividing through by $2n^{d-1}$, taking the superior limit as $n \to \infty$ (n even) and using Theorem 3.5 we obtain

$$\limsup_{n\to\infty, n \text{ even}} \frac{1}{2n^{d-1}} \log Z_{2,n^{(d-1)}}(a) \le (1/2)\log 2 + (1/2) l_{d-1}(a^2/2). \tag{3.10}$$

Relations (3.9) and (3.10) imply the inequality in the statement of the Proposition. □

THEOREM 3.10. *For $a > 0$ we have*

$$\log a \le l_d(a) \le \log a + \frac{1}{2^{d-1}} \log \left(1 + \frac{1}{2}\left(\frac{2}{a}\right)^{2^{d-1}}\right). \tag{3.11}$$

Proof. For the first inequality, notice that for even n there is an admissible orientation of $T_n^{(d)}$ containing only special vertex configurations: just orient all edges so that they point from vertices of one of the bipartition classes to vertices of the other. Since the weight of this orientation is a^{n^d}, we obtain $Z_n^{(d)}(a) \ge a^{n^d}$, which implies the first inequality in (3.11).

To obtain the second inequality, notice that by applying Proposition 3.9 $d-1$ times we obtain

$$l_d(a) \le \left(1 - \frac{1}{2^{d-1}}\right) \log 2 + \frac{1}{2^{d-1}} l_1\left(\frac{a^{2^{d-1}}}{2^{2^{d-1}-1}}\right).$$

However, Lemma 3.7 implies $l_1(b) = \log(1+b)$ for all $b > 0$ and hence we obtain from the above inequality that

$$
\begin{aligned}
l_d(a) &\leq \left(1-\frac{1}{2^{d-1}}\right)\log 2+\frac{1}{2^{d-1}}\log\left(1+2\left(\frac{a}{2}\right)^{2^{d-1}}\right)\\
&=\left(1-\frac{1}{2^{d-1}}\right)\log 2+\frac{1}{2^{d-1}}\log 2\left(\frac{a}{2}\right)^{2^{d-1}}\left(1+\frac{1}{2}\left(\frac{2}{a}\right)^{2^{d-1}}\right)\\
&=\log a+\frac{1}{2^{d-1}}\log\left(1+\frac{1}{2}\left(\frac{2}{a}\right)^{2^{d-1}}\right).
\end{aligned}
$$

□

COROLLARY 3.11. *For $a \geq 2$ we have $\lim_{d\to\infty} l_d(a) = \log a$. In other words, the orientation consisting entirely of special vertex configurations dominates as $d \to \infty$.*

We now return to the problem which motivated the consideration of our d-dimensional model, the study of the number of matchings of the hypergraphs $TD_n^{(d)}$ defined in Section 2.

By (2.1) and Theorem 3.5, the sequence $\{(1/n^d)\log M(TD_n^{(d)})\}_n$ is convergent. Let L_d be its limit.

THEOREM 3.12.

$$
|L_d - \frac{d-1}{2}\log 2| \leq \frac{1}{2^{d+(d-3)2^{d-2}}}. \tag{3.12}
$$

Proof. By (2.1) we obtain that $L_d = l_d(2^{(d-1)/2})$. Apply Theorem 3.10 and use the inequality $\log(1+x) < x$, for $x > 0$. □

REMARK 3.13. The error term in (3.12) decreases very fast as d grows. For $d = 6$ it is already less than 10^{-15}.

REMARK 3.14. Let $Z'_{n^{(d)}}(a)$ be the generating function for $ASP(n^{(d)})$, with pattern A weighted by $a^{N_+(A)+N_-(A)}$. In the correspondence between admissible orientations of $T_n^{(d)}$ and $ASP(n^{(d)})$, the number of admissible orientations having the same ASP is at most 2 to the number of circuits of $T_n^{(d)}$, i.e., $2^{dn^{d-1}}$. When taking the logarithm and dividing by n^d, the contribution of this multiplicative factor approaches zero as $n \to \infty$. Therefore,

$$
\lim_{n\to\infty}(1/n^d)\log Z'_{n^{(d)}}(a) = l_d(a). \tag{3.13}
$$

It is therefore natural to ask whether the set of alternating sign patterns of shape $(n_1, \ldots, n_d)$ satisfies conditions (i)–(iii) from Remark 3.6. For, if this was the case, one could use the arguments above to obtain the statement of Corollary 3.11 for all $a \geq 1$.

However, one can show that this is not the case. Indeed, suppose the alternating sign patterns under consideration did satisfy conditions (i)–(iii) from Remark 3.6. Then all arguments used in this section would go through, and the analogs of Lemma 3.2 and Theorem 3.5 would imply (by taking $a = 1$)

$$
\lim_{n\to\infty}\frac{1}{2^{d-2}n^2}\log|ASP(2^{(d-2)}, n^{(2)})| \leq \lim_{n\to\infty}\frac{1}{2^{d-1}n}\log|ASP(2^{(d-1)}, n)|.
$$

Since $ASP(2^{(k)}, n^{(l)})$ can be identified with $ASP(n^{(l)})$ for all $k, l \geq 1$, the above inequality implies

$$\lim_{n\to\infty} \frac{1}{n^2} \log |ASP(n^{(2)})| \leq \frac{1}{2} \lim_{n\to\infty} \frac{1}{n} \log |ASP(n)|.$$

By (3.13), the limits in the above inequality are equal to $l_2(1)$ and $l_1(1)$, respectively. Then by Lemma 3.7, the right hand side equals $(1/2)\log 2 = 0.34...$. On the other hand, by [**6**] the left hand side equals $(3/2)\log(4/3) = 0.43...$, a contradiction.

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