ENUMERATION OF LOZENGE TILINGS OF PUNCTURED HEXAGONS

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October, 1997

Abstract. We present a combinatorial solution to the problem of determining the number of lozenge tilings of a hexagon with sides a, $b+1$, b, $a+1$, b, $b+1$, with the central unit triangle removed. For $a=b$, this settles an open problem posed by Propp [7].

Let a, b, c be positive integers, and denote by H the hexagon whose side-lengths are (in cyclic order) a, b, c, a, b, c and all whose angles have 120 degrees. The lozenge tilings (i.e., tilings by unit rhombi) of H can be regarded as plane partitions contained in an $a \times b \times c$ box (cf. [**2**]), and therefore their number is given by the simple product formula [**5**]

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}. \tag{1}$$

Motivated by this, Propp [**7**] considered the problem of enumerating the lozenge tilings of a hexagon whose sides are alternately a and $a+1$, from which the central unit triangle has been removed (removal of a suitable unit triangle is necessary for the remaining region to have lozenge tilings). Based on numerical evidence, he conjectured that there exists a simple product formula for the number of tilings of these regions.

The more general question of finding the number of lozenge tilings of a hexagon with sides a, $b+1$, c, $a+1$, b, $c+1$, with the central unit triangle removed – denote it by $N(a,b,c)$ – appeared in work of Kuperberg [**4**] concerning certain weighted enumerations of plane partitions. This general question has been recently settled by Okada and Krattenthaler [**6**], who proved that $N(a,b,c)$ is equal to the product of four factors of type (1) (their proof relies on a new Schur function identity they prove using the minor summation formula of Ishikawa and Wakayama [**3**]).

The purpose of this paper is to give a simple product formula (with a simple combinatorial proof) for $N(a,b,b)$ (this settles in particular Propp's original question; Figure 1 shows the region corresponding to $a=2$, $b=4$).

Let $SC(a,b,c)$ be the number of self-complementary pane partitions that fit in an $a\times b\times c$ box (see [**8**] for the definition). In [**8**] it is given a simple product formula for $SC(a,b,c)$.

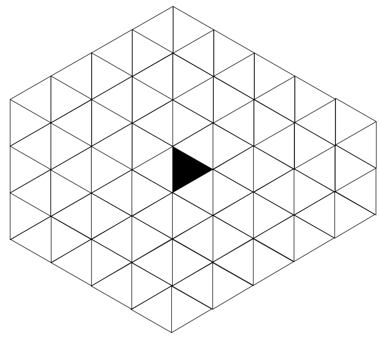

FIGURE 1

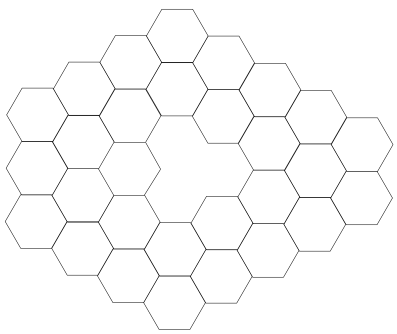

FIGURE 2(a)

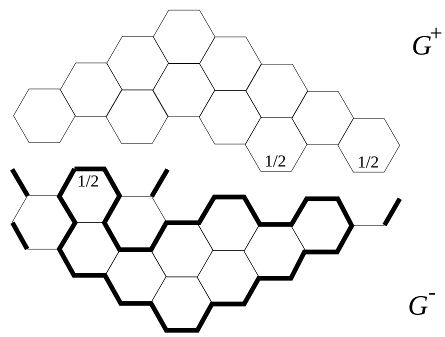

FIGURE 2(b)

THEOREM 1.

$$N(a,b,b) = SC(a+1,b,b)SC(a,b+1,b+1).$$

Proof. Let G be the graph dual to the region of the triangular lattice obtained from a hexagon of size $a \times (b+1) \times b \times (a+1) \times b \times (b+1)$ by removing the central unit triangle (Figure 2(a) illustrates this for $a = 2$, $b = 4$). Any lozenge tiling of our region can be identified with a perfect matching of G. Therefore, $N(a,b,b)$ is just the number $M(G)$ of perfect matchings of G.

The graph G has a symmetry axis; let $v_1, v_2, \dots, v_{2b}$ be the vertices of G on this axis, as they occur from left to right. It is immediate to check that all the conditions in the hypothesis of the Factorization Theorem of [**1**] are met. Applying this to G we obtain that

$$M(G) = 2^b M(G^+)M(G^-), \tag{2}$$

where G^+ (resp., G^-) is the top (resp., bottom) connected component of the subgraph of G obtained by removing the edges incident to the v_i's from above, for $1 \leq i \leq b$, the edges incident to the v_i's from below, for $b+1 \leq i \leq 2b$, and finally by weighting by $1/2$ the edges of these two subgraphs along the symmetry axis of G (see Figure 2(b)).

Consider now the $a \times (b+1) \times (b+1)$ honeycomb graph H (the case a=2, b=4 is pictured in Figure 3(a)); the matchings of this graph are in bijection with the plane partitions fitting in an $a \times (b+1) \times (b+1)$ box. According to this bijection, $SC(a, b+1, b+1)$ is equal to the number of matchings of H that are invariant under rotation by 180 degrees.

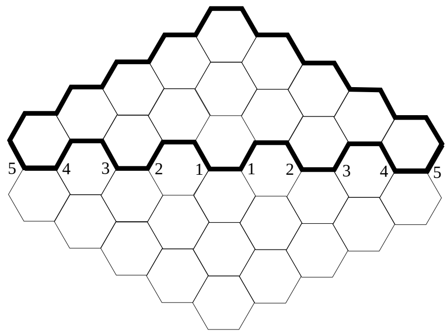

FIGURE 3(a)

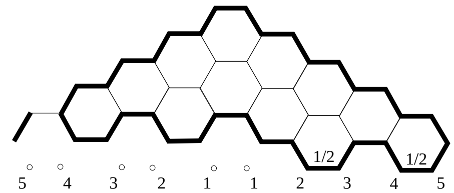

FIGURE 3(b)

Let H_1 be the subgraph of H induced by the vertices on or above its horizontal symmetry axis ℓ (the boundary of H_1 is shown in thick solid lines in Figure 3(a)). Label the vertices of H_1 on ℓ according to their distance to the center of H (the two closest vertices are labeled 1, the next two closest 2, and so on). Denote by H_2 the graph obtained from H_1 by identifying vertices with the same label (if two edges have both endpoints identified they are considered identical; note that the edge whose endpoints are labeled 1 gives rise to a loop). The matchings of H invariant under rotation by 180 degrees can be identified with the matchings of H_2. Therefore,

$$M(H_2) = SC(a, b+1, b+1). \tag{3}$$

The graph H_2 can be symmetrically embedded in the plane. The symmetry axis contains precisely $b+1$ of its vertices. Therefore, if b is even, all perfect matchings of H_2 contain the loop at the vertex labeled 1 (henceforth referred to simply as the loop), while for odd b none of them contains it.

Suppose b is even (the case b odd is treated similarly). Since all matchings of H_2 contain the loop, we may remove it (together with the vertex labeled 1) without changing the number of matchings of our graph; for the sake of notational simplicity, denote the resulting graph still by H_2.

Even though H_2 is not "separated" by its symmetry axis in the sense of [**1**], the variant of the Factorization Theorem in [**1**, Section 7] is applicable and yields

$$M(H_2) = 2^{b/2} M(H_3), \tag{4}$$

where H_3 is the graph obtained from H_1 by removing the edges incident from above to the leftmost $b+2$ vertices on ℓ and then weighting the edges along ℓ of the remaining subgraph by 1/2. However, remarkably, the graph obtained from H_3 by removing the vertices matched by forced edges is isomorphic to G^+ (see Figure 3(b)). We obtain therefore from (3) and (4) that

$$M(G^+) = 2^{-b/2} SC(a, b+1, b+1). \tag{5}$$

To determine $M(G^-)$, take H to be the $(a+1) \times b \times b$ honeycomb graph. Construct the graphs H_1 and H_2 as before (see Figure 4(a)). Since the symmetry axis of H_2 contains

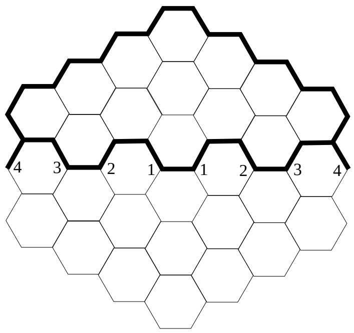

FIGURE 4(a)

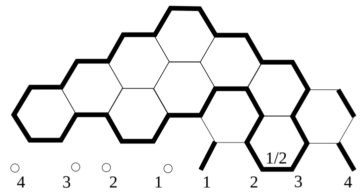

FIGURE 4(b)

now b vertices (and b is even), no perfect matching of H_2 contains the loop, and therefore we may replace H_2 by its subgraph obtained by removing this loop (and *keeping* the vertex labeled 1). Applying the variant of the Factorization Theorem in [**1**, Section 7] we obtain

$$M(H_2) = 2^{b/2} M(H_3), \tag{6}$$

where H_3 is the graph obtained from H_1 by removing the edges incident from above to the leftmost b vertices on ℓ and then weighting the edges along ℓ of the remaining subgraph by $1/2$. However, again, the graph obtained from H_3 by removing the vertices matched by forced edges is isomorphic to the subgraph of G^- left after removing its vertices matched by forced edges (see Figure 4(b)). Since now $M(H_2) = SC(a+1, b, b)$, (6) implies

$$M(G^-) = 2^{-b/2} SC(a+1, b, b). \tag{7}$$

The statement of the theorem follows from (2), (5) and (7).

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