Math 1553 Worksheet: More lines, planes, §1.1, and §1.2, Spring 2018

Solutions

1. Find all values of h so that the lines $x+hy=-5$ and $2x-8y=6$ do *not* intersect.

Solution.

We can use standard algebra, row operations, or geometric intuition.

Using standard algebra: Let's see what happens when the lines *do* intersect. In that case, there is a point (x,y) where

$$x+hy=-5$$
$$2x-8y=6.$$

Subtracting twice the first equation from the second equation gives us

$$x+hy=-5$$
$$0+(-8-2h)y=16.$$

If $-8-2h=0$ (so $h=-4$), then the second line is $0\cdot y=16$, which is impossible. In other words, if $h=-4$ then we cannot find a solution to the system of two equations, so the two lines *do not* intersect.

On the other hand, if $h\neq -4$, then we can solve for y above:

$$(-8-2h)y=16 \qquad y=\frac{16}{-8-2h} \qquad y=\frac{8}{-4-h}.$$

We can now substitute this value of y into the first equation to find x:

$$x+hy=-5 \qquad x+h\cdot\frac{8}{-4-h}=-5 \qquad x=-5-\frac{8h}{-4-h}.$$

Therefore, the lines fail to intersect if and only if $\boxed{h=-4}$.

Using row operations: Like the previous technique, let's see what happens if the lines intersect. We put the equations into augmented matrix form and use row operations.

$$\left[\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right] \xrightarrow{R_2=R_2-2R_1} \left[\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right].$$

If $-8-2h=0$ (so $h=-4$), then the second equation is $0=16$, so our system has no solutions. In other words, the lines do not intersect.

If $h\neq -4$, then the second equation is $(-8-2h)y=16$, so $y=\dfrac{16}{-8-2h}=\dfrac{8}{-4-h}$, and $x=-5-hy=-5-\dfrac{8h}{-4-h}$, so the lines intersect at $\left(-5-\dfrac{8h}{-4-h},\ \dfrac{8}{-4-h}\right)$.

Therefore, our answer is $h=-4$.

Using intuition from geometry: Two non-identical lines in the xy-plane intersect if and only if they are not parallel. The second line is $y = \frac{1}{4}x - \frac{3}{4}$, so its slope is $\frac{1}{4}$. If $h \neq 0$, then the first line is $y = -\frac{1}{h}x - \frac{5}{h}$, so the lines are parallel when $-\frac{1}{h} = \frac{1}{4}$, which means $h = -4$. You can check that when $h = -4$ the lines aren't identical. (And if $h = 0$ then the first line is vertical so it isn't parallel to the second).

2. **a)** Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

b) Which entries are the pivots? Which are the pivot columns?

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \begin{pmatrix} 1 & 4 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution.

Both matrices are in reduced row echelon form. The pivots are in red; the other entries in the pivot columns are in blue.

3. Suppose that each augmented matrix below represents a linear system in the variables x, y, and z (with the last column being after the $=$ sign). Which of the systems are consistent? Which have a *unique* solution?

Solution.

(a) $$\left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 5R_1} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -4} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + 8R_2} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{array}\right)$$

The bottom row says $0 = -10$, which is preposterous. Therefore, the system is inconsistent (it has no solutions).

(b) $$\left(\begin{array}{ccc|c} 3 & -4 & 2 & 0 \\ -8 & 12 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 + 3R_1} \left(\begin{array}{ccc|c} 3 & -4 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 3 & -4 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + 6R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 8 & 11 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -4} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 8 & 11 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 8R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 \div 3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This is the reduced row echelon form. It says that the system has the unique solution

$$\boxed{x = 0, \quad y = 0, \quad z = 0}.$$

Alternatively, we could have used back-substitution at the time we reached

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right).$$

The third row gives $3z = 0 \implies z = 0$.
The second row now gives $y + 0 \implies y = 0$.
The first row gives $x + 2(0) = 0 \implies x = 0$.