

Section 2.8

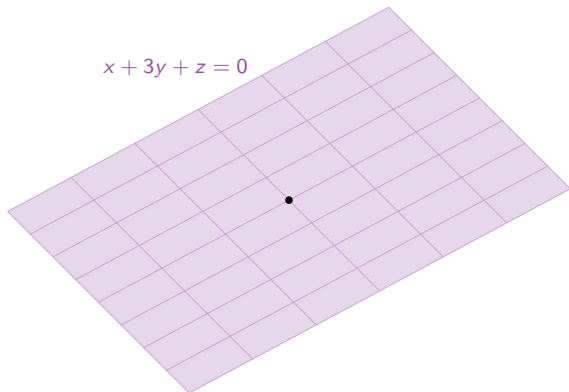
Subspaces of \mathbf{R}^n

Motivation

Today we will discuss **subspaces** of \mathbf{R}^n .

A subspace turns out to be the same as a span, except we don't know *which* vectors it's the span of.

This arises naturally when you have, say, a plane through the origin in \mathbf{R}^3 which is *not* defined (a priori) as a span, but you still want to say something about it.



Definition of Subspace

Definition

A **subspace** of \mathbf{R}^n is a subset V of \mathbf{R}^n satisfying:

1. The zero vector is in V . "not empty"
2. If u and v are in V , then $u + v$ is also in V . "closed under addition"
3. If u is in V and c is in \mathbf{R} , then cu is in V . "closed under \times scalars"

Fast-forward

Every subspace is a span, and every span is a subspace.

A subspace is a span of some vectors, but you haven't computed what those vectors are yet.

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What does this mean?

- ▶ If v is in V , then all scalar multiples of v are in V by (3). That is, the line through v is in V .
- ▶ If u, v are in V , then xu and yv are in V for scalars x, y by (3). So $xu + yv$ is in V by (2). So $\text{Span}\{u, v\}$ is contained in V .
- ▶ Likewise, if v_1, v_2, \dots, v_n are all in V , then $\text{Span}\{v_1, v_2, \dots, v_n\}$ is contained in V : a subspace contains the span of any set of vectors in it.

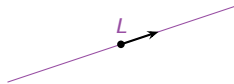
If you pick enough vectors in V , eventually their span will fill up V , so:

A subspace is a span of some set of vectors in it.

Examples

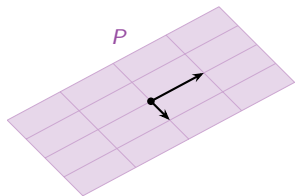
Example

A line L through the origin: this contains the span of any vector in L .



Example

A plane P through the origin: this contains the span of any vectors in P .



Example

All of \mathbf{R}^n : this contains 0 , and is closed under addition and scalar multiplication.

Example

The subset $\{0\}$: this subspace contains only one vector.

Note these are all pictures of spans! (Line, plane, space, etc.)

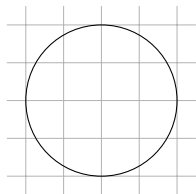
Subsets and Subspaces

They aren't the same thing

A **subset** of \mathbf{R}^n is any collection of vectors whatsoever.

All of the following non-examples are still subsets.

A **subspace** is a special kind of subset, which satisfies the three defining properties.



Subset: *yes*

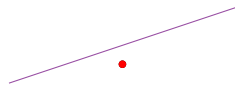
Subspace: *no*

Non-Examples

Non-Example

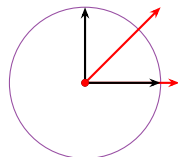
A line L (or any other set) that doesn't contain the origin is not a subspace.

Fails: 1.



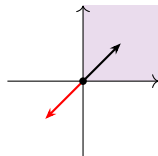
Non-Example

A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."



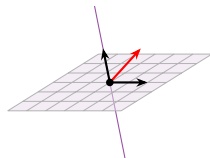
Non-Example

The first quadrant in \mathbf{R}^2 is not a subspace. Fails: 3 only.



Non-Example

A line union a plane in \mathbf{R}^3 is not a subspace. Fails: 2 only.



Spans are Subspaces

Theorem

Any Span $\{v_1, v_2, \dots, v_n\}$ is a subspace.

!!!

Every subspace is a span, and every span is a subspace.

Definition

If $V = \text{Span}\{v_1, v_2, \dots, v_n\}$, we say that V is the subspace **generated by** or **spanned by** the vectors v_1, v_2, \dots, v_n .

Check:

1. $0 = 0v_1 + 0v_2 + \dots + 0v_n$ is in the span.
2. If, say, $u = 3v_1 + 4v_2$ and $v = -v_1 - 2v_2$, then

$$u + v = 3v_1 + 4v_2 - v_1 - 2v_2 = 2v_1 + 2v_2$$

is also in the span.

3. Similarly, if u is in the span, then so is cu for any scalar c .

Poll

Which of the following are subspaces?

- A. The empty set $\{\}$.
- B. The solution set to a homogeneous system of linear equations.
- C. The solution set to an inhomogeneous system of linear equations.
- D. The set of all vectors in \mathbf{R}^n with rational (fraction) coordinates.

For the ones which are not subspaces, which property(ies) do they not satisfy?

- A. This is not a subspace: it does not contain the zero vector.
- B. This is a subspace: the solution set is a span, produced by finding the parametric vector form of the solution.
- C. This is not a subspace: it does not contain 0.
- D. This is not a subspace: it is not closed under multiplication by scalars (e.g. by π).

Subspaces

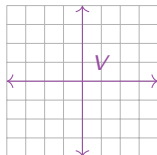
Verification

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \mid ab = 0 \right\}$. Let's check if V is a subspace or not.

1. Does V contain the zero vector? $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies ab = 0$ ✓

3. Is V closed under scalar multiplication?

- ▶ Let $\begin{pmatrix} a \\ b \end{pmatrix}$ be (an unknown vector) in V .
- ▶ *This means:* a and b are numbers such that $ab = 0$.
- ▶ Let c be a scalar. Is $c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}$ in V ?
- ▶ *This means:* $(ca)(cb) = 0$.
- ▶ Well, $(ca)(cb) = c^2(ab) = c^2(0) = 0$ ✓



2. Is V closed under addition?

- ▶ Let $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} a' \\ b' \end{pmatrix}$ be (unknown vectors) in V .
- ▶ *This means:* $ab = 0$, and $a'b' = 0$.
- ▶ Is $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \end{pmatrix}$ in V ?
- ▶ *This means:* $(a+a')(b+b') = 0$.
- ▶ This is not true for all such a, a', b, b' : for instance, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are in V , but their sum $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in V , because $1 \cdot 1 \neq 0$. ✗

We conclude that V is *not* a subspace. A picture is above. (It doesn't look like a span.)

Column Space and Null Space

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

- ▶ The **column space** of A is the subspace of \mathbf{R}^m spanned by the columns of A . It is written $\text{Col } A$.
- ▶ The **null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$:

$$\text{Nul } A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

This is a subspace of \mathbf{R}^n .

The column space is defined as a span, so we know it is a subspace. It is the range (as opposed to the codomain) of the transformation $T(x) = Ax$.

Check that the null space is a subspace:

1. 0 is in $\text{Nul } A$ because $A0 = 0$.
2. If u and v are in $\text{Nul } A$, then $Au = 0$ and $Av = 0$. Hence

$$A(u + v) = Au + Av = 0,$$

so $u + v$ is in $\text{Nul } A$.

3. If u is in $\text{Nul } A$, then $Au = 0$. For any scalar c , $A(cu) = cAu = 0$. So cu is in $\text{Nul } A$.

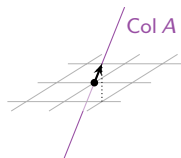
Column Space and Null Space

Example

$$\text{Let } A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let's compute the column space:

$$\text{Col } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$



This is a line in \mathbf{R}^3 .

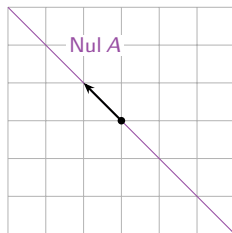
Let's compute the null space:

The reduced row echelon form of A is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This gives the equation $x + y = 0$, or

$$\begin{array}{l} x = -y \\ y = y \end{array} \quad \begin{array}{l} \text{parametric vector form} \\ \text{~~~~~} \end{array} \quad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence the null space is $\text{Span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$, a line in \mathbf{R}^2 .



The Null Space is a Span

The column space of a matrix A is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

Question

How to find vectors which span the null space?

Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{if, say, } x_3 \text{ and } x_4 \text{ are the free variables. So} \quad \text{Nul } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §1.5 (Solution Sets).

Note: It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

How do you check if a subset is a subspace?

- ▶ Is it a span? Can it be written as a span?
- ▶ Can it be written as the column space of a matrix?
- ▶ Can it be written as the null space of a matrix?
- ▶ Is it all of \mathbf{R}^n or the zero subspace $\{0\}$?
- ▶ Can it be written as a type of subspace that we'll learn about later (eigenspaces, ...)?

If so, then it's automatically a subspace.

If all else fails:

- ▶ Can you verify directly that it satisfies the three defining properties?

What is the *smallest number* of vectors that are needed to span a subspace?

Definition

Let V be a subspace of \mathbf{R}^n . A **basis** of V is a set of vectors $\{v_1, v_2, \dots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \dots, v_m\}$, and
2. $\{v_1, v_2, \dots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V , and is written $\dim V$.

Note the big
red border here

Why is a basis the smallest number of vectors needed to span?

Recall: *linearly independent* means that every time you add another vector, the span gets bigger.

Hence, if we remove any vector, the span gets *smaller*: so any smaller set can't span V .

Important

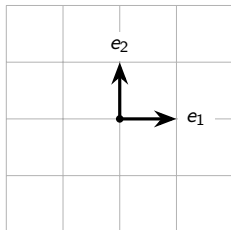
A subspace has *many different* bases, but they all have the same number of vectors (see the exercises in §2.9).

Question

What is a basis for \mathbf{R}^2 ?

We need two vectors that *span* \mathbf{R}^2 and are *linearly independent*. $\{e_1, e_2\}$ is one basis.

1. They span: $\begin{pmatrix} a \\ b \end{pmatrix} = ae_1 + be_2$.
2. They are linearly independent because they are not collinear.

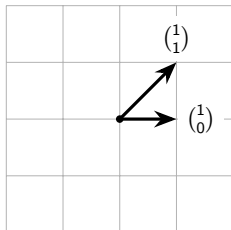


Question

What is another basis for \mathbf{R}^2 ?

Any two nonzero vectors that are not collinear. $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is also a basis.

1. They span: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every row.
2. They are linearly independent: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every column.



The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \dots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

are a basis for \mathbf{R}^n .  The identity matrix has columns e_1, e_2, \dots, e_n .

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general: $\{v_1, v_2, \dots, v_n\}$ is a basis for \mathbf{R}^n if and only if the matrix

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}$$

has a pivot in every row and every column, i.e. if A is *invertible*.

Sanity check: we have shown that $\dim \mathbf{R}^n = n$.

Basis of a Subspace

Example

Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x + 3y + z = 0 \right\} \quad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that \mathcal{B} is a basis for V . (So $\dim V = 2$: it is a *plane*.) [\[interactive\]](#)

0. **In V :** both vectors are in V because

$$-3 + 3(1) + 0 = 0 \quad \text{and} \quad 0 + 3(1) + (-3) = 0.$$

1. **Span:** If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in V , then $y = -\frac{1}{3}(x + z)$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{x}{3} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - \frac{z}{3} \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

2. **Linearly independent:**

$$c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} -3c_1 \\ c_1 + c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0.$$

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A .

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric vector form}} x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis of Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

1. The vectors span Nul A by construction (every solution to $Ax = 0$ has this form).
2. Can you see why they are linearly independent? (Look at the last two rows.)

Fact

The *pivot columns* of A always form a basis for Col A .

Warning: I mean the pivot columns of the *original* matrix A , not the row-reduced form. (Row reduction changes the column space.)

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis \leftarrow pivot columns in rref

So a basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

Why? See slides on linear independence.

Summary

- ▶ A **subspace** is the same as a span of some number of vectors, but we haven't computed the vectors yet.
- ▶ To any matrix is associated two subspaces, the **column space** and the **null space**:

$\text{Col } A =$ the span of the columns of A

$\text{Nul } A =$ the solution set of $Ax = 0$.

- ▶ A **basis** of a subspace is a minimal set of spanning vectors; the **dimension** of V is the number of vectors in any basis.
- ▶ The pivot columns form a basis for $\text{Col } A$, and the parametric vector form produces a basis for $\text{Nul } A$.

Warning

These are not the official definitions!