

Section 5.3

Diagonalization

Motivation

Difference equations

Many real-word linear algebra problems have the form:

$$v_1 = Av_0, \quad v_2 = Av_1 = A^2v_0, \quad v_3 = Av_2 = A^3v_0, \quad \dots \quad v_n = Av_{n-1} = A^nv_0.$$

This is called a **difference equation**.

Our toy example about rabbit populations had this form.

The question is, what happens to v_n as $n \rightarrow \infty$?

- ▶ Taking powers of diagonal matrices is easy!
- ▶ Taking powers of *diagonalizable* matrices is still easy!
- ▶ Diagonalizing a matrix is an eigenvalue problem.

Powers of Diagonal Matrices

If D is diagonal, then D^n is also diagonal; its diagonal entries are the n th powers of the diagonal entries of D :

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^3 = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}, \quad \dots \quad D^n = \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}.$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad D^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix},$$
$$\dots \quad D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{pmatrix}$$

Powers of Matrices that are Similar to Diagonal Ones

What if A is not diagonal?

Example

Let $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$. Compute A^n .

In §5.2 lecture we saw that A is similar to a diagonal matrix:

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$

$$A^3 = (PDP^{-1})(PD^2P^{-1}) = PD(P^{-1}P)D^2P^{-1} = PDID^2P^{-1} = PD^3P^{-1}$$

\vdots

$$A^n = PD^nP^{-1}$$

Closed formula in terms of n :
easy to compute



Therefore

$$A^n = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix} \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2^n + (-1)^n & 2^n + (-1)^{n+1} \\ 2^n + (-1)^{n+1} & 2^n + (-1)^n \end{pmatrix}.$$

Diagonalizable Matrices

Definition

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = PDP^{-1} \quad \text{for } D \text{ diagonal.}$$

Important

$$\text{If } A = PDP^{-1} \text{ for } D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix} \text{ then}$$

$$A^k = PD^k P^{-1} = P \begin{pmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{pmatrix} P^{-1}.$$

So diagonalizable matrices are easy to raise to any power.

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In this case, $A = PDP^{-1}$ for

$$P = \left(\begin{array}{c|c|c|c} | & | & \cdots & | \\ \hline v_1 & v_2 & \cdots & v_n \\ \hline | & | & & | \end{array} \right) \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where v_1, v_2, \dots, v_n are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Corollary  a theorem that follows easily from another theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

The Corollary is true because eigenvectors with distinct eigenvalues are always linearly independent. We will see later that a diagonalizable matrix need not have n distinct eigenvalues though.

Diagonalization

Easy example

Question: What does the Diagonalization Theorem say about the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}?$$

This is a triangular matrix, so the eigenvalues are the diagonal entries 1, 2, 3.

A diagonal matrix just scales the coordinates by the diagonal entries, so we can take our eigenvectors to be the unit coordinate vectors e_1, e_2, e_3 . Hence the Diagonalization Theorem says

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It doesn't give us anything new because the matrix was already diagonal!

A diagonal matrix D is diagonalizable! It is similar to itself:

$$D = I_n D I_n^{-1}.$$

Diagonalization

Example

Problem: Diagonalize $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2).$$

Therefore the eigenvalues are -1 and 2 . Let's compute some eigenvectors:

$$(A + I)x = 0 \iff \begin{pmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = -y$, so $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue -1 .

$$(A - 2I)x = 0 \iff \begin{pmatrix} -3/2 & 3/2 \\ 3/2 & -3/2 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = y$, so $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 2 .

The eigenvectors v_1, v_2 are linearly independent, so the Diagonalization Theorem says

$$A = PDP^{-1} \quad \text{for} \quad P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Diagonalization

Another example

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda - 2).$$

Therefore the eigenvalues are 1 and 2, with respective multiplicities 2 and 1.

Let's compute the 1-eigenspace:

$$(A - I)x = 0 \iff \begin{pmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

The parametric vector form is

$$\begin{array}{l} x = y \\ y = y \\ z = z \end{array} \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence a basis for the 1-eigenspace is

$$\mathcal{B}_1 = \{v_1, v_2\} \quad \text{where} \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Diagonalization

Another example, continued

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

Now let's compute the 2-eigenspace:

$$(A - 2I)x = 0 \iff \begin{pmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = 3z, y = 2z$, so an eigenvector with eigenvalue 2 is

$$v_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

The eigenvectors v_1, v_2, v_3 are linearly independent: v_1, v_2 form a basis for the 1-eigenspace, and v_3 is not contained in the 1-eigenspace. Therefore the Diagonalization Theorem says

$$A = PDP^{-1} \quad \text{for} \quad P = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note: In this case, there are three linearly independent eigenvectors, but only two distinct eigenvalues.

Diagonalization

A non-diagonalizable matrix

Problem: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

This is an upper-triangular matrix, so the only eigenvalue is 1. Let's compute the 1-eigenspace:

$$(A - I)x = 0 \iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x = 0.$$

This is row reduced, but has only one free variable x ; a basis for the 1-eigenspace is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. So *all eigenvectors* of A are multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Conclusion: A has only one linearly independent eigenvector, so by the “only if” part of the diagonalization theorem, A is not diagonalizable.

Poll

Which of the following matrices are diagonalizable, and why?

$$\text{A. } \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{B. } \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

Matrix **A** is not diagonalizable: its only eigenvalue is 1, and its 1-eigenspace is spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Matrix **B** is diagonalizable because it is a 2×2 matrix with distinct eigenvalues.

Diagonalization

Procedure

How to diagonalize a matrix A :

1. Find the eigenvalues of A using the characteristic polynomial.
2. For each eigenvalue λ of A , compute a basis \mathcal{B}_λ for the λ -eigenspace.
3. If there are fewer than n total vectors in the union of all of the eigenspace bases \mathcal{B}_λ , then the matrix is not diagonalizable.
4. Otherwise, the n vectors v_1, v_2, \dots, v_n in your eigenspace bases are linearly independent, and $A = PDP^{-1}$ for

$$P = \left(\begin{array}{c|c|c|c} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{array} \right) \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue for v_i .

Diagonalization

Proof

Why is the Diagonalization Theorem true?

A diagonalizable implies A has n linearly independent eigenvectors: Suppose $A = PDP^{-1}$, where D is diagonal with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. Let v_1, v_2, \dots, v_n be the columns of P . They are linearly independent because P is invertible. So $Pe_i = v_i$, hence $P^{-1}v_i = e_i$.

$$Av_i = PDP^{-1}v_i = PDe_i = P(\lambda_i e_i) = \lambda_i Pe_i = \lambda_i v_i.$$

Hence v_i is an eigenvector of A with eigenvalue λ_i . So the columns of P form n linearly independent eigenvectors of A , and the diagonal entries of D are the eigenvalues.

A has n linearly independent eigenvectors implies A is diagonalizable: Suppose A has n linearly independent eigenvectors v_1, v_2, \dots, v_n , with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Let P be the invertible matrix with columns v_1, v_2, \dots, v_n . Let $D = P^{-1}AP$.

$$De_i = P^{-1}APe_i = P^{-1}Av_i = P^{-1}(\lambda_i v_i) = \lambda_i P^{-1}v_i = \lambda_i e_i.$$

Hence D is diagonal, with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. Solving $D = P^{-1}AP$ for A gives $A = PDP^{-1}$.

Non-Distinct Eigenvalues

Definition

Let λ be an eigenvalue of a square matrix A . The **geometric multiplicity** of λ is the dimension of the λ -eigenspace.

Theorem

Let λ be an eigenvalue of a square matrix A . Then

$$1 \leq (\text{the geometric multiplicity of } \lambda) \leq (\text{the algebraic multiplicity of } \lambda).$$

The proof is beyond the scope of this course.

Corollary

Let λ be an eigenvalue of a square matrix A . If the algebraic multiplicity of λ is 1, then the geometric multiplicity is also 1.

The Diagonalization Theorem (Alternate Form)

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of A equals n .
3. The sum of the algebraic multiplicities of the eigenvalues of A equals n , and *the geometric multiplicity equals the algebraic multiplicity* of each eigenvalue.

Non-Distinct Eigenvalues

Examples

Example

If A has n distinct eigenvalues, then the algebraic multiplicity of each equals 1, hence so does the geometric multiplicity, and therefore A is diagonalizable.

For example, $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$ has eigenvalues -1 and 2 , so it is diagonalizable.

Example

The matrix $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -(\lambda - 1)^2(\lambda - 2).$$

The algebraic multiplicities of 1 and 2 are 2 and 1, respectively. They sum to 3. We showed before that the geometric multiplicity of 1 is 2 (the 1-eigenspace has dimension 2). The eigenvalue 2 automatically has geometric multiplicity 1. Hence the geometric multiplicities add up to 3, so A is diagonalizable.

Non-Distinct Eigenvalues

Another example

Example

The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial $f(\lambda) = (\lambda - 1)^2$.

It has one eigenvalue 1 of algebraic multiplicity 2.

We showed before that the geometric multiplicity of 1 is 1 (the 1-eigenspace has dimension 1).

Since the geometric multiplicity is smaller than the algebraic multiplicity, the matrix is *not* diagonalizable.

Applications to Difference Equations

$$\text{Let } D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Fix a vector v_0 , and let $v_1 = Dv_0$, $v_2 = Dv_1$, etc., so $v_n = D^n v_0$.

Question: What happens to the v_i 's for different choices of v_0 ?

Answer: Note that D is diagonal, so

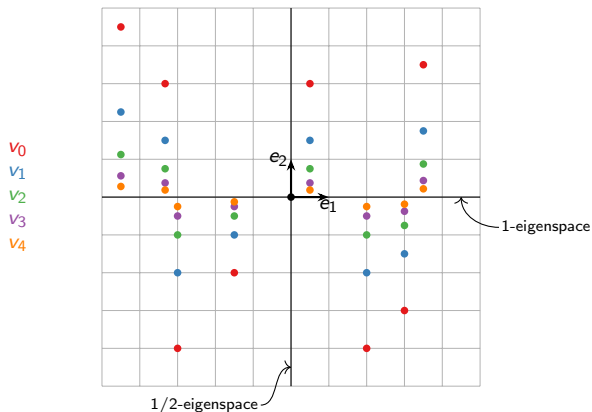
$$D^n \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1^n & 0 \\ 0 & 1/2^n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b/2^n \end{pmatrix}.$$

So the x -coordinate of v_n equals the x -coordinate of v_0 , and the y -coordinate gets halved every time.

Applications to Difference Equations

Picture

$$D \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b/2 \end{pmatrix}$$



So all vectors get “sucked into the x -axis,” which is the 1-eigenspace.

Applications to Difference Equations

More complicated example

$$\text{Let } A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}.$$

Fix a vector v_0 , and let $v_1 = Av_0$, $v_2 = Av_1$, etc., so $v_n = A^n v_0$.

Question: What happens to the v_i 's for different choices of v_0 ?

Answer: We want to compute powers of A , so this is a diagonalization question. The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right).$$

We compute eigenvectors with eigenvalues 1 and 1/2 to be, respectively,

$$w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad w_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, $A = PDP^{-1}$ for $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$.

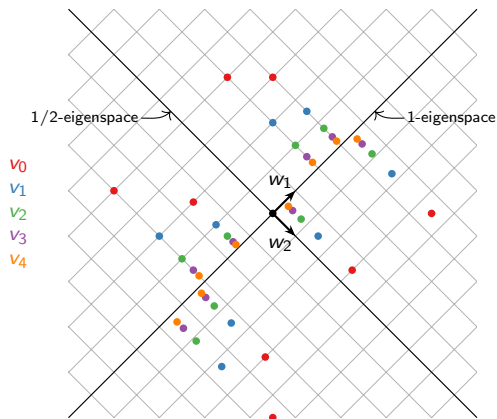
This is the same matrix D from before. Hence

$$v_n = A^n v_0 = PD^n P^{-1} v_0.$$

Applications to Difference Equations

Picture of the more complicated example

Recall: $A^n = PD^nP^{-1}$ acts on the \mathcal{B} -coordinates in the same way that D^n acts on the usual coordinates, where $\mathcal{B} = \{w_1, w_2\}$.



So all vectors get “sucked into the 1-eigenspace.”

[interactive]

Applications to Difference Equations

Remark

The matrix $A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$ is called a **stochastic matrix**.

- ▶ A matrix A is **diagonalizable** if it is similar to a diagonal matrix D :
 $A = PDP^{-1}$.
- ▶ It is easy to take powers of diagonalizable matrices: $A^r = PD^rP^{-1}$.
- ▶ An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors v_1, v_2, \dots, v_n , in which case $A = PDP^{-1}$ for

$$P = \left(\begin{array}{c|c|ccc|c} | & | & & & | \\ \hline v_1 & v_2 & \cdots & & v_n \\ \hline | & | & & & | \end{array} \right) \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

- ▶ If A has n distinct eigenvalues, then it is diagonalizable.
- ▶ The **geometric multiplicity** of an eigenvalue λ is the dimension of the λ -eigenspace.
- ▶ $1 \leq (\text{geometric multiplicity}) \leq (\text{algebraic multiplicity})$.
- ▶ An $n \times n$ matrix is diagonalizable if and only if the sum of the geometric multiplicities is n .