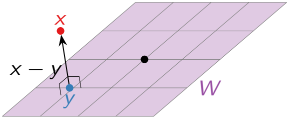
Section 6.4

The Gram–Schmidt Process

Motivation: Best Approximation

Suppose you measure a data point x which you know for theoretical reasons must lie on a subspace W.



Due to measurement error, though, the measured x is not actually in W. Best approximation: y is the *closest* point to x on W.

How do you know that y is the closest point? The vector from y to x is orthogonal to W: it is in the *orthogonal complement* $W^\perp$.

Note $x = y + (x - y)$, where y is in W and $x - y$ is in $W^\perp$. Last time we called this the *orthogonal decomposition* of x:

$$x = x_W + x_{W^\perp} \qquad x_W = y \qquad x_{W^\perp} = x - y.$$

Recall: If W is a subspace of $\mathbf{R}^n$, its **orthogonal complement** is

$$W^\perp = \{v \text{ in } \mathbf{R}^n \mid v \text{ is perpendicular to every vector in } W\}$$

Theorem

Every vector x in $\mathbf{R}^n$ can be written as

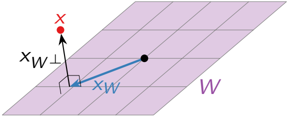
$$x = x_W + x_{W^\perp}$$

for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.

The equation $x = x_W + x_{W^\perp}$ is called the **orthogonal decomposition** of x (with respect to W).

The vector x_W *is the closest vector to* x *on* W.

[interactive 1] [interactive 2]



How do you compute x_W? (Note $x_{W^\perp} = x - x_W$.)

Recall: a set of nonzero vectors $\{u_1, u_2, \ldots, u_m\}$ is **orthogonal** if $u_i \cdot u_j = 0$ when $i \neq j$: each vector is perpendicular to the others.

Definition

Let W be a subspace of $\mathbf{R}^n$, and let $\{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\mathrm{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_n}{u_n \cdot u_n}\, u_n.$$

[interactive]

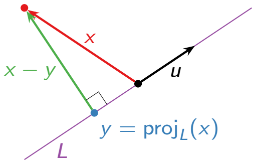
Let x be a vector and let $x = x_W + x_{W^\perp}$ be its orthogonal decomposition with respect to a subspace W. The following vectors are the same:

- x_W
- $\mathrm{proj}_W(x)$
- The closest vector to x on W

The formula for orthogonal projections is simple when W is a *line*.

Let $L = \mathrm{Span}\{u\}$ be a line in $\mathbf{R}^n$, and let x be in $\mathbf{R}^n$. The orthogonal projection of x onto L is the point

$$\mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u}\, u.$$



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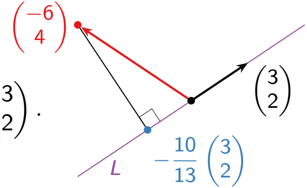
Example: Compute the orthogonal projection of $x = \binom{-6}{4}$ onto the line L spanned by $u = \binom{3}{2}$.

$$y = \mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u = \frac{-18+8}{9+4}\begin{pmatrix}3\\2\end{pmatrix} = -\frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix}.$$



We can think of orthogonal projection as a *transformation*:

$$\text{proj}_W \colon \mathbf{R}^n \longrightarrow \mathbf{R}^n \qquad x \mapsto \text{proj}_W(x).$$

Theorem

Let W be a subspace of $\mathbf{R}^n$.

1. proj_W is a *linear* transformation.
2. For every x in W, we have $\text{proj}_W(x) = x$.
3. For every x in $W^\perp$, we have $\text{proj}_W(x) = 0$.
4. The range of proj_W is W and the null space of proj_W is $W^\perp$.

Let W be a subspace with orthogonal basis $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$.

For x in W we have $\text{proj}_W(x) = x$, so

$$x = \text{proj}_W(x) = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_n}{u_n \cdot u_n}\, u_n$$

$$\implies [x]_{\mathcal{B}} = \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \ldots, \frac{x \cdot u_m}{u_m \cdot u_m} \right).$$

[interactive]

A Non-Orthogonal Basis

Important: Orthogonal projections require an *orthogonal* basis!

Non-Example: Consider the basis $\mathcal{B} = \{v_1, v_2\}$ of $\mathbf{R}^2$, where

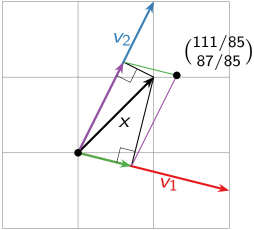
$$v_1 = \begin{pmatrix} 2 \\ -1/2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This is not orthogonal: $\binom{2}{-1/2} \cdot \binom{1}{2} = 1 \neq 0$.

Let's try to compute $x = \mathrm{proj}_{\mathbf{R}^2}(x)$ for $x = \binom{1}{1}$ using the basis $\{v_1, v_2\}$:

$$x = \mathrm{proj}_{\mathbf{R}^2}(x) = \frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{3/2}{17/4} \begin{pmatrix} 2 \\ -1/2 \end{pmatrix} + \frac{3}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 111/85 \\ 87/85 \end{pmatrix}$$ ✗

This does not work!



[interactive] (compare [orthogonal basis])

Recap

All of the procedures we learned in §§6.2–6.3 require an *orthogonal* basis $\{u_1, u_2, \ldots, u_m\}$.

- Finding the orthogonal projection of a vector x onto the span W of $u_1, u_2, \ldots, u_m$:
$$\text{proj}_W(x) = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i.$$
- Finding the orthogonal decomposition of x:
$$x = \text{proj}_W(x) + x_{W^\perp}.$$
- Finding the $\mathcal{B}$-coordinates of x:
$$[x]_{\mathcal{B}} = \left(\frac{x \cdot u_1}{u_1 \cdot u_1},\ \frac{x \cdot u_2}{u_2 \cdot u_2},\ \ldots,\ \frac{x \cdot u_m}{u_m \cdot u_m} \right).$$

Problem: What if your basis isn't orthogonal?

Solution: The Gram–Schmidt process: take any basis and make it orthogonal.

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

1. $u_1 = v_1$
2. $u_2 = v_2 - \text{proj}_{\text{Span}\{u_1\}}(v_2) \qquad = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1$
3. $u_3 = v_3 - \text{proj}_{\text{Span}\{u_1, u_2\}}(v_3) \qquad = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$

$\vdots$

m. $u_m = v_m - \text{proj}_{\text{Span}\{u_1, u_2, \ldots, u_{m-1}\}}(v_m) = v_m - \displaystyle\sum_{i=1}^{m-1} \frac{v_m \cdot u_i}{u_i \cdot u_i}\, u_i$

Then $\{u_1, u_2, \ldots, u_m\}$ is an *orthogonal* basis for the same subspace W.

Remark

In fact, for every i between 1 and m, the set $\{u_1, u_2, \ldots, u_i\}$ is an orthogonal basis for $\text{Span}\{v_1, v_2, \ldots, v_i\}$.

Find an orthogonal basis $\{u_1, u_2\}$ for $W = \mathrm{Span}\{v_1, v_2\}$, where

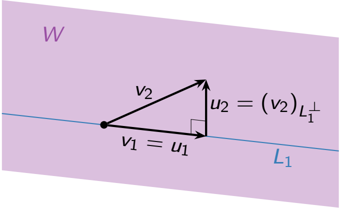
$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Run Gram–Schmidt:

1. $u_1 = v_1$ 2. $u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$

Why does this work?

- First we take $u_1 = v_1$.
- Now we're sad because $u_1 \cdot v_2 \neq 0$, so we can't take $u_2 = v_2$.
- Fix: let $L_1 = \mathrm{Span}\{u_1\}$, and let $u_2 = (v_2)_{L_1^\perp} = v_2 - \mathrm{proj}_{L_1}(v_2)$.
- By construction, $u_1 \cdot u_2 = 0$, because $L_1 \perp u_2$.



Important: $\mathrm{Span}\{u_1, u_2\} = \mathrm{Span}\{v_1, v_2\} = W$: this is an *orthogonal* basis for the *same* subspace.

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \text{Span}\{v_1, v_2, v_3\} = \mathbf{R}^3$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

Run Gram–Schmidt:

1. $u_1 = v_1$
2. $u_2 = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \dfrac{2}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
3. $u_3 = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$
   $= \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \dfrac{4}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \dfrac{1}{1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Important: $\text{Span}\{u_1, u_2, u_3\} = \text{Span}\{v_1, v_2, v_3\} = W$: this is an *orthogonal* basis for the *same* subspace.

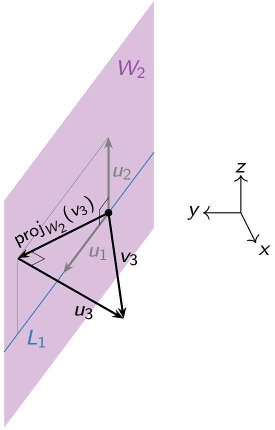
The Gram–Schmidt Process

Three vectors, continued

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix},\ v_3 = \begin{pmatrix}3\\1\\1\end{pmatrix} \overset{\text{G–S}}{\rightsquigarrow} u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ u_2 = \begin{pmatrix}0\\0\\1\end{pmatrix},\ u_3 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

Why does this work?

- Once we have u_1 and u_2, then we're sad because v_3 is not orthogonal to u_1 and u_2.
- Fix: let $W_2 = \mathrm{Span}\{u_1, u_2\}$, and let $u_3 = (v_3)_{W_2^\perp} = v_3 - \mathrm{proj}_{W_3}(u_3)$.
- By construction, $u_1 \cdot u_3 = 0 = u_2 \cdot u_3$ because $W_2 \perp u_3$.



Check:

$$u_1 \cdot u_2 = 0\ ✓$$
$$u_1 \cdot u_3 = 0\ ✓$$
$$u_2 \cdot u_3 = 0\ ✓$$

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \text{Span}\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 4 \\ -2 \\ -2 \\ 0 \end{pmatrix}.$$

Run Gram–Schmidt:

1. $u_1 = v_1$
2. $u_2 = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} - \dfrac{6}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{pmatrix}$
3. $u_3 = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$

$$= \begin{pmatrix} 4 \\ -2 \\ -2 \\ 0 \end{pmatrix} - \frac{0}{24} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-20}{25} \begin{pmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -2 \end{pmatrix}$$

Poll

Poll

What happens if you try to run Gram–Schmidt on a linearly *dependent* set of vectors $\{v_1, v_2, \ldots, v_m\}$?

A. You get an inconsistent equation.

B. For some i you get $u_i = u_{i-1}$.

C. For some i you get $u_i = 0$.

D. You create a rift in the space-time continuum.

If $\{v_1, v_2, \ldots, v_m\}$ is linearly dependent, then some v_i is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}$.

This means

$$v_i = \mathrm{proj}_{\mathrm{Span}\{u_1,u_2,\ldots,u_{i-1}\}}(v_i)$$

$$\implies u_i = v_i - \mathrm{proj}_{\mathrm{Span}\{u_1,u_2,\ldots,u_{i-1}\}}(v_i) = 0.$$

In this case, you can simply discard u_i and v_i and continue: so Gram–Schmidt produces an orthogonal basis from any spanning set!

Summary

- We like orthogonal bases because they let us compute orthogonal projections.
- The Gram–Schmidt process turns an arbitrary basis into an orthogonal basis.