**Math 1553 Supplement, §1.2 and §1.3**

Solutions

1. Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 1 & -3 & -4 & -9 \\ 3 & -1 & 8 & 9 \end{array}\right)$$

**Question:** Does the corresponding linear system have a solution? If so, what is the solution set?

a) Formulate this question as a vector equation.

b) Formulate this question as a system of linear equations.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

f) Find a **different** solution in parts (e) and (d).

**Solution.**

a) What are the solutions to the following vector equation?

$$x\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + z\begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ 9 \end{pmatrix}$$

b) What is the solution set of the following linear system?

$$\begin{array}{rcr} 2x - 2y + 2z = & 0 \\ x - 3y - 4z = & -9 \\ 3x - y + 8z = & 9 \end{array}$$

c) There exists a solution if and only if $\begin{pmatrix} 0 \\ -9 \\ 9 \end{pmatrix}$ is in $\mathrm{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}\right\}$.

e) Row reducing yields

$$\left(\begin{array}{ccc|c} 1 & 0 & 7/2 & 9/2 \\ 0 & 1 & 5/2 & 9/2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Hence $z$ is a free variable, so the solution in parametric form is

$$\begin{array}{l} x = \frac{9}{2} - \frac{7}{2}z \\ y = \frac{9}{2} - \frac{5}{2}z. \end{array}$$

Taking $z = 0$ yields the solution $x = y = 9/2$.

f) Taking $z = 1$ yields the solution $x = 1, y = 2$.

**2.** Let $v_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$    $v_2 = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$    $w = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}$.

**Question:** Is $w$ a linear combination of $v_1$ and $v_2$? In other words, is $w$ in Span$\{v_1, v_2\}$?

**a)** Formulate this question as a vector equation.

**b)** Formulate this question as a system of linear equations.

**c)** Formulate this question as an augmented matrix.

**d)** Answer the question using the interactive demo.

**e)** Answer the question using row reduction.

## Solution.

**a)** Does the following vector equation have a solution?

$$x \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}$$

**b)** Does the following linear system have a solution?

$$\begin{aligned} 2x - 2y &= 2 \\ x - 3y &= -4 \\ 3x - y &= 8 \end{aligned}$$

**c)** As an augmented matrix:

$$\left( \begin{array}{cc|c} 2 & -2 & 2 \\ 1 & -3 & -4 \\ 3 & -1 & 8 \end{array} \right)$$

**e)** Row reducing yields

$$\left( \begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{array} \right)$$

so $x = 7/2$ and $y = 5/2$.

**3.** Let

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

Is $b$ in the span of the columns of $A$? In other words, is $b$ a linear combination of the columns of $A$? Justify your answer.

### Solution.

Let $v_1$, $v_2$, and $v_3$ be the columns of $A$. We are asked to determine whether there are scalars $x_1$, $x_2$, and $x_3$ so that $x_1 v_1 + x_2 v_2 + x_3 v_3 = b$, which means

$$\begin{aligned} x_1 \qquad\quad + 5x_3 &= \phantom{-}2 \\ -2x_1 + \phantom{2}x_2 - 6x_3 &= -1 \\ 2x_2 + 8x_3 &= \phantom{-}6 \end{aligned}$$

We translate the system of linear equations into an augmented matrix, and row reduce it:

$$\left( \begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array} \right) \xrightarrow{\text{rref}} \left( \begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The right column is not a pivot column, so the system is consistent. Therefore, $b$ is in the span of the columns of $A$ (in other words, $b$ is a linear combination of the columns of $A$).

We weren't asked to solve the equation explicitly, but if we wanted to do so, we would use the RREF of the matrix above to write

$$x_1 = 2 - 5x_3 \qquad x_2 = 3 - 4x_3 \qquad x_3 = x_3 \ (x_3 \text{ is free}).$$

In fact, we can take $x_1 = 2, x_2 = 3$, and $x_3 = 0$, to write

$$b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}.$$

**4.** Consider the vector equation

$$x \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + z \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}.$$

**Question:** Is there a solution? If so, what is the solution set?

   **a)** Formulate this question as an augmented matrix.

   **b)** Formulate this question as a system of linear equations.

   **c)** What does this mean in terms of spans?

   **d)** Answer the question using the interactive demo.

   **e)** Answer the question using row reduction.

## Solution.

**a)** As an augmented matrix:

$$\left( \begin{array}{ccc|c} 2 & -2 & 3 & -5 \\ 1 & -1 & 0 & -1 \\ 3 & -1 & 4 & -2 \end{array} \right)$$

**b)** What is the solution set of the following linear system?

$$\begin{aligned} 2x - 2y + 3z &= -5 \\ x - y \phantom{{}+ 3z} &= -1 \\ 3x - y + 4z &= -2 \end{aligned}$$

**c)** There exists a solution if and only if $\begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}$ is in Span $\left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right\}$.

**e)** Row reducing yields

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -1 \end{array} \right),$$

so $x = 3/2$, $y = 5/2$, and $z = -1$.

**5.** Decide if each of the following statements is true or false. If it is true, prove it; if it is false, provide a counterexample.

    **a)** Every set of four or more vectors in $\mathbf{R}^3$ will span $\mathbf{R}^3$.

    **b)** The span of any set contains the zero vector.

## Solution.

    **a)** This is **false.** For instance, the vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \right\}$$

only span the $x$-axis.

    **b)** This is **true.** We have

$$0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_p.$$

Aside: the span of the empty set is equal to $\{0\}$, because 0 is the empty sum, i.e. the sum with no summands. Indeed, if you add the empty sum to a vector $v$, you get $v + $ (no other summands), which is just $v$; and the only vector which gives you $v$ when you add it to $v$, is 0. (If you find this argument intriguing, you might want to consider taking abstract math courses later on.)