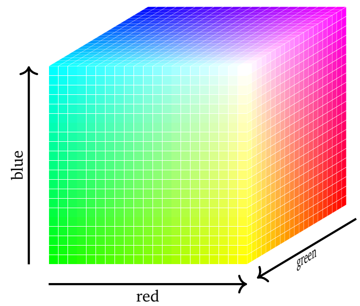
Math 1553 Worksheet §1.7, 1.8, 1.9

Solutions

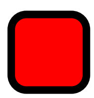
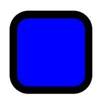
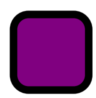
1. Every color on my computer monitor is a vector in $\mathbf{R}^3$ with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.



Given colors $v_1, v_2, \ldots, v_p$, we can form a "weighted average" of these colors by making a linear combination

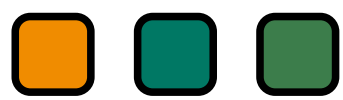
$$v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

with $c_1 + c_2 + \cdots + c_p = 1$. Example:

$\frac{1}{2}$  $+$ $\frac{1}{2}$  $=$ 

Consider the colors on the right. Are these colors linearly independent? What does this tell you about the colors?
After doing this problem, check out the interactive demo, where you can adjust sliders to find a prescribed color.

$$\begin{pmatrix}240\\140\\0\end{pmatrix} \quad \begin{pmatrix}0\\120\\100\end{pmatrix} \quad \begin{pmatrix}60\\125\\75\end{pmatrix}$$



Solution.

The vectors

$$\begin{pmatrix}240\\140\\0\end{pmatrix}, \quad \begin{pmatrix}0\\120\\100\end{pmatrix}, \quad \begin{pmatrix}60\\125\\75\end{pmatrix}$$

are linearly independent if and only if the vector equation

$$x\begin{pmatrix}240\\140\\0\end{pmatrix} + y\begin{pmatrix}0\\120\\100\end{pmatrix} + z\begin{pmatrix}60\\125\\75\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

has only the trivial solution. This translates into the matrix (we don't need to augment since it's a homogeneous system)

$$\begin{pmatrix} 240 & 0 & 60 \\ 140 & 120 & 125 \\ 0 & 100 & 75 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & .25 \\ 0 & 1 & .75 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{parametric}} \begin{aligned} x &= -.25z \\ y &= -.75z \end{aligned}$$

Hence the vectors are linearly *de*pendent; taking $z = 1$ gives the linear dependence relation

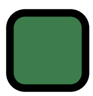
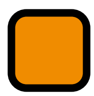
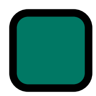
$$-\frac{1}{4}\begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix} + \begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Rearranging gives

$$\begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix}.$$

In terms of colors:

$$\begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix}$$

2. The second little pig has decided to build his house out of sticks. The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z-axis (look downward onto the xy-plane the way we usually picture the plane as $\mathbf{R}^2$), and then projected onto the xy-plane. Find the matrix for this transformation.

Solution.

To compute the matrix for T, we have to compute $T(e_1), T(e_2)$, and $T(e_3)$. To see the picture, let's put ourselves above the xy-plane (with the usual orientation of the x and y axes in the xy-plane), looking downward. For e_1 and e_2, it is as if we are applying $\begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then putting a zero in the z-coordinate each time. We find

$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Rotating e_3 around the z-axis does nothing, and projecting onto the xy-plane sends it to zero, so $T(e_3) = 0$. Therefore, the matrix for T is

$$\begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

a) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (z, x)$.

b) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution.

a) This is onto. If (a, b) is any vector in the codomain $\mathbf{R}^2$, then $(a, b) = T(b, 0, a)$, so (a, b) is in the range. It is not one-to-one though: indeed, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$. Alternatively, we could have computed the matrix for T and found that it has a pivot in every row but not in every column, which would have led us to the same conclusion.

b) This matrix is already row reduced. We can see that does not have a pivot in every row *or* in every column, so it is neither onto nor one-to-one. In fact, if $T(x) = Ax$ then

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_3 \\ 0 \end{pmatrix},$$

so we can see that $(0, 0, 1)$ is not in the range of T, and that

$$T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = T\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$