Math 1553 Supplement §5.3 (with some more practice from §5.2)

For those who want additional practice problems after completing the worksheet, here are some extra practice problems.

1. Let A and B be 3×3 real matrices. Answer yes / no / maybe:

a) If A and B have the same eigenvalues, then A is similar to B.

b) If A is diagonalizable and invertible, then A^{-1} is diagonalizable.

c) If A and B are invertible and A is similar to B, then A^{-1} is similar to B^{-1}.

Solution.

a) Maybe. For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ have the same eigenvalues ($\lambda = 0$ with alg. multiplicity 2) but are not similar, whereas $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is similar to itself.

b) Yes. If $A = PDP^{-1}$ and A is invertible then its eigenvalues are all nonzero, so the diagonal entries of D are nonzero and thus D is invertible (pivot in every diagonal position). Thus, $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$.

c) Yes. $A = PBP^{-1}$ for some invertible P, so

$$A^{-1} = (PBP^{-1})^{-1} = (P^{-1})^{-1}B^{-1}P^{-1} = PB^{-1}P^{-1}.$$

2. Let $A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix}$.

The characteristic polynomial for A is $-\lambda^3 + 7\lambda^2 - 16\lambda + 12$, and $\lambda - 3$ is a factor. Decide if A is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution.

By polynomial division,

$$\frac{-\lambda^3 + 7\lambda^2 - 16\lambda + 12}{\lambda - 3} = -\lambda^2 + 4\lambda - 4 = -(\lambda-2)^2.$$

Thus, the characteristic poly factors as $-(\lambda-3)(\lambda-2)^2$, so the eigenalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

For $\lambda_1 = 3$, we row-reduce $A - 3I$:

$$\begin{pmatrix} 5 & 36 & 62 \\ -6 & -37 & -62 \\ 3 & 18 & 30 \end{pmatrix} \xrightarrow[\text{(New } R_1)/3]{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 6 & 10 \\ -6 & -37 & -62 \\ 5 & 36 & 62 \end{pmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & 6 & 10 \\ 0 & -1 & -2 \\ 0 & 6 & 12 \end{pmatrix}$$

$$\xrightarrow[\text{then } R_2 = -R_2]{R_3 = R_3 + 6R_2} \begin{pmatrix} 1 & 6 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 - 6R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the solutions to $\left(A-3I \mid 0\right)$ are $x_1 = 2x_3,\ \ x_2 = -2x_3,\ \ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}. \qquad \text{The 3-eigenspace has basis } \left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda_2 = 2$, we row-reduce $A-2I$:

$$\begin{pmatrix} 6 & 36 & 62 \\ -6 & -36 & -62 \\ 3 & 18 & 31 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 6 & \frac{31}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solutions to $\left(A-2I \quad 0\right)$ are $x_1 = -6x_2 - \frac{31}{3}x_3,\ x_2 = x_2,\ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6x_2 - \frac{31}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix}.$$

The 2-eigenspace has basis $\left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$.

Therefore, $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 2 & -6 & -\frac{31}{3} \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that we arranged the eigenvectors in P in order of the eigenvalues $3,\ 2,\ 2$, so we had to put the diagonals of D in the same order.

3. Give an example of a non-diagonal 2×2 matrix which is diagonalizable but not invertible. Justify your answer.

Solution.

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not invertible (row of zeros) but is diagonalizable since its has two distinct eigenvalues 0 and 1 (it is triangular, so its diagonals are its eigenvalues)