MATH 1553
FINAL EXAMINATION, SPRING 2018

Name	

Circle the name of your instructor below:

Fathi — Jankowski, lecture A — Jankowski, lecture C — Kordek

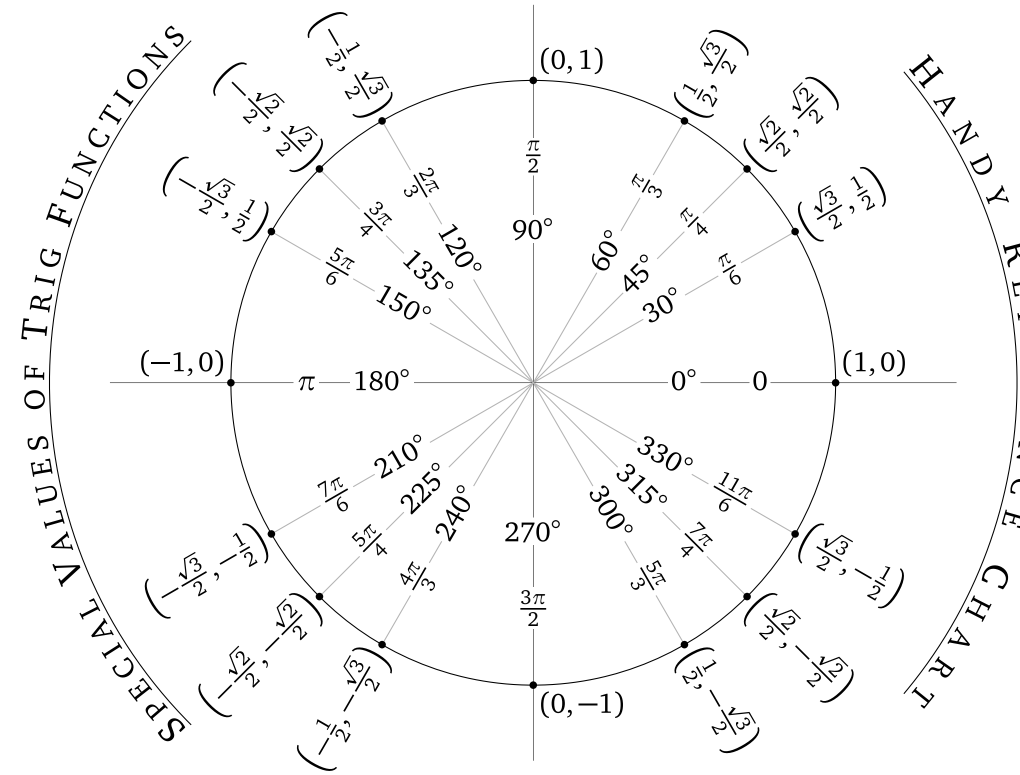
Strenner, lecture H — Strenner, lecture M — Yan

DO NOT WRITE IN THE TABLE BELOW! It will be used to record scores.

1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (notes, text, etc.).
- Please show your work. A correct answer without appropriate work will receive little or no credit.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!


Special Values of Trig Functions
Handy Reference Chart
(0, 1)
(1, 0)
(−1, 0)
(0, −1)
0°
0
30°
π/6
45°
π/4
60°
π/3
90°
π/2
120°
2π/3
135°
3π/4
150°
5π/6
180°
π
210°
7π/6
225°
5π/4
240°
4π/3
270°
3π/2
300°
5π/3
315°
7π/4
330°
11π/6
(√3/2, 1/2)
(√2/2, √2/2)
(1/2, √3/2)
(−1/2, √3/2)
(−√2/2, √2/2)
(−√3/2, 1/2)
(−√3/2, −1/2)
(−√2/2, −√2/2)
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(√3/2, −1/2)

Problem 1. [2 points each]

True or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**.
You do not need to justify your answer, and there is no partial credit.
In each case, assume that the entries of all matrices and all vectors are real numbers.

a) **T** **F** Let A be an $n \times n$ matrix. If A has two identical columns, then A is not invertible.

b) **T** **F** If A is a 3×4 matrix and b is in $\mathbf{R}^3$, then the set of solutions to $Ax = b$ is a subspace of $\mathbf{R}^4$.

c) **T** **F** If A is a 3×7 matrix then $\text{rank}(A) < \dim(\text{Nul } A)$.

d) **T** **F** If A is an $n \times n$ matrix with n linearly independent eigenvectors, then each eigenvalue of A has algebraic multiplicity 1.

e) **T** **F** If A and B are 2×2 matrices that both have λ as an eigenvalue, then λ^2 is an eigenvalue of AB.

f) **T** **F** If v and w are nonzero orthogonal vectors, then $\text{proj}_{\text{Span}\{v\}} w$ is the zero vector.

g) **T** **F** The least-squares solution to $Ax = b$ is unique if

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

h) **T** **F** If A is a 4×3 matrix and $\text{Col } A$ is 2-dimensional, then the orthogonal complement of $\text{Col } A$ is also 2-dimensional.

Solution.

a) True. Having two identical columns guarantees that A has linearly dependent columns, hence A is not invertible.

b) False. If the system is inconsistent then it has no solutions. Even if the system has a solution, the set of solutions won't be a subspace if $b \neq 0$ since it won't include the zero vector.

c) True. By the Rank Theorem we know $\text{rank}(A) + \dim(\text{Nul } A) = 7$. Since Col A is a subspace of $\mathbf{R}^3$ we know $\text{rank}(A) \leq 3$, so $\dim(\text{Nul } A)$ is at least 4.

d) False. Take $A = I_3$ for example, then A has 3 linearly independent eigenvectors but its only eigenvalue is $\lambda = 1$ which has algebraic multiplicity 3.

e) False. $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ both have $\lambda = 2$ as an eigenvalue, but $AB = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ so AB does not have 4 as an eigenvalue.

f) True, since $\text{proj}_{\text{Span}\, v}\, w = \dfrac{w \cdot v}{v \cdot v} v = \dfrac{0}{||v||^2} v = 0$.

g) False. The equation $A^T A \widehat{x} = A^T b$ is $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \widehat{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which has infinitely many solutions. Alternatively, we can see that there will be infinitely solutions by observing that the columns of A are linearly dependent.

h) True. Since Col A lives in $\mathbf{R}^4$, the orthogonal complement formula gives $\dim(\text{Col}A) + \dim((\text{Col}A)^\perp) = 4$, so $\dim((\text{Col}A)^\perp) = 2$.

Problem 2.

[10 points]

Short answer. On this page, you do not need to show your work. There is no partial credit for (a), (b), or (c).

a) Find $(AB)^{-1}$ if $A^{-1} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix}$.

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 15 & 5 \end{pmatrix}.$$

b) Which of the following are subspaces of $\mathbf{R}^3$? Circle all that apply.

(i) The plane $x - y + z = 1$ in $\mathbf{R}^3$.

(ii) The z-axis in $\mathbf{R}^3$.

(iii) The set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ that satisfy $x + 3y = z$.

c) Write a nonzero 2×2 matrix A which is upper-triangular and satisfies $A^2 = 0$.

Answer: any matrix of the form $A = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$ where c is a nonzero real number.

d) Write three different 3×3 matrices A, B, and C which each have eigenvalue $\lambda = -1$ with algebraic multiplicity 3, so that no two of the different matrices are similar.

The (-1)-eigenspaces must have different dimensions for each matrix. Below, the dimension of the (-1)-eigenspace is 3 for A, 2 for B, and 1 for C.

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Problem 3. [10 points]

Short answer. Show your computations for credit on (b) and (c).

a) Let u and v be orthogonal vectors in $\mathbf{R}^3$ with $||u|| = 6$ and $||v|| = 1$.
Find $u \cdot (u - v)$.

$u \cdot (u + v) = u \cdot u - u \cdot v = ||u||^2 + 0 = 36.$

b) Find a nonzero vector v orthogonal to $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \\ 8 \end{pmatrix}$.

We put the vectors as rows of a matrix A and find its nullspace.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -3 & 1 & 8 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 + 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 11 & 0 \end{array}\right), \qquad x_1 = -x_3 \qquad x_2 = -11x_3 \qquad x_3 = x_3.$$

One such vector is $v = \begin{pmatrix} -1 \\ -11 \\ 1 \end{pmatrix}$. In fact, any nonzero multiple of $\begin{pmatrix} -1 \\ -11 \\ 1 \end{pmatrix}$ is an answer to this problem.

c) Use row reduction to find the inverse of the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The RREF of $(M|I)$ is

$$\left(M \mid I \right) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 = R_2 - R_3]{R_1 = R_1 - R_3} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

so $M^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

d) In the following questions, b_1 and b_2 are vectors in $\mathbf{R}^3$.
Which statements are possible? Circle all that apply.

(i) b_1 and b_2 are nonzero and orthogonal, but the set $\{b_1, b_2\}$ is linearly dependent.

(ii) $\{b_1, b_2\}$ is a linearly independent set, but b_1 and b_2 are not orthogonal.

Problem 4. [10 points]

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that rotates counterclockwise by $\dfrac{\pi}{6}$ radians, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects about the line $y = x$.

a) Find the standard matrix A for T and the standard matrix B for U.

$$A = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

b) Find the matrix for T^{-1} and the matrix for U^{-1}. Clearly label your answers.

Recall the formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

For T^{-1}: $A^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. For U^{-1}: $B^{-1} = \dfrac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(alternatively, A^{-1} is just *clockwise* rotation by $\pi/3$ radians)

c) Compute the matrix M for the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ that first rotates *clockwise* by $\frac{\pi}{6}$ radians, then reflects about the line $y = x$, then rotates counterclockwise by $\frac{\pi}{6}$ radians.

This is the transformation that first does T^{-1}, then does U, then does T. In other words, we want the transformation for $(T \circ U \circ T^{-1})$.

$$\begin{aligned} M = ABA^{-1} &= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}. \end{aligned}$$

Problem 5.

[8 points]

Consider the following matrix A, and its reduced row echelon form.

$$A = \begin{pmatrix} 1 & 0 & -3 & 1 \\ -2 & 3 & -6 & 4 \\ -1 & 4 & -13 & 7 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Find a basis for Col A.

The first two columns are pivot columns, so a basis for Col A is $\left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \right\}$.

In fact, no two columns of A are collinear, so any two columns of A will form basis for Col A. However, using any number of columns of the RREF of A will give the wrong answer.

b) Find a basis for Nul A.

The RREF of A gives us the equations

$$x_1 = 3x_3 - x_4, \qquad x_2 = 4x_3 - 2x_4, \qquad x_3 = x_3, \qquad x_4 = x_4.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_3 - x_4 \\ 4x_3 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 3 \\ 4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

c) What is $\dim\big((\mathrm{Nul} A)^{\perp}\big)$? Briefly justify your answer.

Since Nul$A$ is a subspace of $\mathbf{R}^4$, $\dim(\mathrm{Nul}\ A) + \dim((\mathrm{Nul}A)^{\perp}) = 4$, so

$$\dim((\mathrm{Nul}A)^{\perp}) = 4 - 2 = 2.$$

Problem 6.

[9 points]

Parts (a) and (b) are unrelated.

a) Compute the determinant of A, where $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 3 \\ 0 & -2 & 1 & 0 \end{pmatrix}$.

We could use row-reduction or cofactors.
Cofactors: Expand along the 4th column to get

$$\det(A) = 3(-1)^7 \det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 1 \end{pmatrix} = -3\det\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = (-3)(-1) = 3.$$

Row-reduction:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 3 \\ 0 & -2 & 1 & 0 \end{pmatrix} \xrightarrow{R_3=R_3-R_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 0 & 3 \\ 0 & -2 & 1 & 0 \end{pmatrix} \xrightarrow[R_4=R_4+2R_2]{R_3=R_3+2R_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{R_4=R_4-R_3/2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & \frac{-3}{2} \end{pmatrix}$$

This matrix has the same determinant as A since every step was a row replacement, so $\det(A) = 1 \cdot 1 \cdot -2 \cdot (-\frac{3}{2}) = 3$.

b) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -8 \\ 0 \\ -2 \end{pmatrix} \right\}$. Find an orthogonal basis for W.

$$u_1 = v_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \end{pmatrix}.$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \end{pmatrix} - \frac{-6}{6} \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 4 \end{pmatrix}.$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{pmatrix} 4 \\ -8 \\ 0 \\ -2 \end{pmatrix} - \frac{12}{6} \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \end{pmatrix} - 0 = \begin{pmatrix} 4 \\ -8 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 4 \\ -2 \end{pmatrix}.$$

Problem 7.

[10 points]

Consider the matrix $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

a) Compute the characteristic polynomial of A.

b) Write the eigenvalues of A.

c) For each eigenvalue of A, compute a basis for the corresponding eigenspace.

d) Decide whether A is diagonalizable. If it is diagonalizable, find an invertible 3×3 matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If A is not diagonalizable, explain why.

Solution.

a) $\det(A - \lambda I) = (1-\lambda)(\lambda^2 - 1) = (1-\lambda)(\lambda-1)(\lambda+1) = -(\lambda-1)^2(\lambda+1)$
(any of these forms is fine)

b) The roots of the characteristic polynomial are $\lambda = 1$ and $\lambda = -1$.

c) For $\lambda = 1$: $\left(A - \lambda I \mid 0\right) = \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 + R_1} \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

So $x_1 = x_1$, $x_2 = x_3$, and $x_3 = x_3$. A basis for the 1-eigenspace is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

For $\lambda = -1$: $\left(A - \lambda I \mid 0\right) = \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_1 = R_1 - R_2]{R_3 = R_3 - R_2} \left(\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 = R_1/2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

So $x_1 = x_3$, $x_2 = -x_3$, $x_3 = x_3$. A basis for the (-1)-eigenspace is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

d) The matrix A has three linearly independent eigenvectors, so it is diagonalizable. Many examples are possible for P and D, but the student match each eigenvector with its corresponding eigenvalue.

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Problem 8. [10 points]

Let $A = \begin{pmatrix} 2 & -6 \\ 2 & 2 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.
(b) Find the complex eigenvalues of A. Fully simplify your answer.
(c) For the eigenvalue with negative imaginary part, find a corresponding eigenvector.
(d) Find a matrix C that represents a composition of scaling and rotation and is similar to A.
(e) What is the scale factor for C?

Solution.

(a) The characteristic polynomial of A is given by

$$\det\begin{pmatrix} 2-\lambda & -6 \\ 2 & 2-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda)+12 = 4-4\lambda+\lambda^2+12 = \lambda^2-4\lambda+16.$$

(b) $\lambda = \dfrac{4 \pm \sqrt{16-64}}{2} = \dfrac{4 \pm \sqrt{-48}}{2} = \dfrac{4 \pm 4\sqrt{3}i}{2} = 2 \pm 2\sqrt{3}i$

(c) For $\lambda = 2-2\sqrt{3}$, we have

$$\left(A-\lambda I \mid 0\right) = \left(\begin{array}{cc|c} 2-(2-2\sqrt{3}i) & -6 & 0 \\ (*) & (*) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 2\sqrt{3}i & -6 & 0 \\ (*) & (*) & 0 \end{array}\right)$$

so an eigenvector is $v = \begin{pmatrix} 6 \\ 2\sqrt{3}i \end{pmatrix}$. Other answers are possible.

For example, $v = \begin{pmatrix} -6 \\ -2\sqrt{3}i \end{pmatrix}$ is also an eigenvector, and so is $v = \begin{pmatrix} -i\sqrt{3} \\ 1 \end{pmatrix}$.

(d) We can use the formula $C = \begin{pmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix}$ for either eigenvalue.

For $\lambda = 2-2\sqrt{3}i$ we will get $C = \begin{pmatrix} 2 & -2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}$.

For $\lambda = 2+2\sqrt{3}i$ we will get $C = \begin{pmatrix} 2 & 2\sqrt{3} \\ -2\sqrt{3} & 2 \end{pmatrix}$.

(e) The scale factor is $|\lambda| = \sqrt{2^2+(2\sqrt{3})^2} = \sqrt{4+12} = 4$.

Problem 9.

[9 points]

Let $W = \text{Span}\{v_1, v_2\}$, where $v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

a) Find the closest point w in W to $x = \begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix}$.

The closest point is

$$w = \text{proj}_W\, x = \frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{28-4}{6}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \frac{-8}{8}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = 4\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -4-2 \\ 8-0 \\ 4-2 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \\ 2 \end{pmatrix}.$$

b) Find the distance from w to $\begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix}$.

$$||x - w|| = \left\| \begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} -6 \\ 8 \\ 2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix} \right\| = \sqrt{36+36+36} = \sqrt{108}.$$

(it is also fine if the student simplifies $\sqrt{108}$ to $6\sqrt{3}$)

c) Find the standard matrix A for the orthogonal projection onto $\text{Span}\{v_1\}$.

$$\text{proj}_{\text{Span}\{v_1\}}\, e_1 = \frac{e_1 \cdot v_1}{v_1 \cdot v_1} v_1 = \frac{-1}{6}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/6 \\ -1/3 \\ -1/6 \end{pmatrix} \quad \text{proj}_{\text{Span}\{v_1\}}\, e_2 = \frac{e_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \frac{2}{6}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \\ 1/3 \end{pmatrix}.$$

$$\text{proj}_{\text{Span}\{v_1\}}\, e_3 = \frac{e_3 \cdot v_1}{v_1 \cdot v_1} v_1 = \frac{1}{6}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/6 \\ 1/3 \\ 1/6 \end{pmatrix}.$$

$$A = \begin{pmatrix} \text{proj}_{\text{Span}\{v_1\}}\, e_1 & \text{proj}_{\text{Span}\{v_1\}}\, e_2 & \text{proj}_{\text{Span}\{v_1\}}\, e_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}.$$

Problem 10. [8 points]

Find the best-fit line $y = c + mx$ for the points $(-5, -6)$, $(-2, 9)$, and $(1, 12)$.

Solution.

If such a line fit the points exactly, we would have

$$-6 = c - 5m$$
$$9 = c - 2m$$
$$12 = c + m$$

This is the system $Ax = b$ where $A = \begin{pmatrix} 1 & -5 \\ 1 & -2 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -6 \\ 9 \\ 12 \end{pmatrix}$. To find the best-fit line, we use least-squares:

$$(A^T A)\widehat{x} = A^T b$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -5 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c \\ m \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -5 & -2 & 1 \end{pmatrix}\begin{pmatrix} -6 \\ 9 \\ 12 \end{pmatrix}.$$

$$\begin{pmatrix} 3 & -6 \\ -6 & 30 \end{pmatrix}\begin{pmatrix} c \\ m \end{pmatrix} = \begin{pmatrix} 15 \\ 24 \end{pmatrix}.$$

$$\left(\begin{array}{cc|c} 3 & -6 & 15 \\ -6 & 30 & 24 \end{array}\right) \xrightarrow{R_2 = R_2 + 2R_1} \left(\begin{array}{cc|c} 3 & -6 & 15 \\ 0 & 18 & 54 \end{array}\right) \xrightarrow[R_2 = R_2/18]{R_1 = R_1/3} \left(\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1 = R_1 + 2R_2} \left(\begin{array}{cc|c} 1 & 0 & 11 \\ 0 & 1 & 3 \end{array}\right).$$

Thus $c = 11$ and $m = 3$.

$$y = 11 + 3x.$$

Scratch paper. This sheet will not be graded under any circumstances.