Math 1553: Final Exam Extra Practice Problems

Spring 2018

These problems are additional extra practice for the final. They are not meant to be 100% comprehensive in scope, and they tend to be more computational than conceptual.

1. Define the following terms: span, linear combination, linearly independent, linear transformation, column space, null space, transpose, inverse, dimension, rank, determinant, eigenvalue, eigenvector, eigenspace, diagonalizable, orthogonal, orthonormal.

2. Let A be an $m \times n$ matrix.

a) How do you determine the pivot columns of A?

b) What do the pivot columns tell you about the equation $Ax = b$?

c) What space is equal to the span of the pivot columns?

d) What is the difference between solving $Ax = b$ and $Ax = 0$? How are the two solution sets related geometrically?

e) If $\text{rank}(A) = r$, where $0 \leq r \leq n$, then how many columns have pivots? What is the dimension of the null space?

Solution.

a) Do row operations until A is in a row echelon form. The leading entries of the rows are the pivots.

b) If there is a pivot in every column, then $Ax = b$ has zero or one solution. Otherwise, $Ax = b$ has zero or infinitely many solutions.

c) The pivot columns form a basis for the column space $\text{Col}A$.

d) Suppose that $Ax = b$ has some solution x_0. Then every other solution to $Ax = b$ has the form $x_0 + x$, where x is a solution to $Ax = 0$. In other words, the solution set to $Ax = b$ is either empty, or it is a translate of the solution set to $Ax = 0$ (the null space).

e) If $\text{rank}(A) = r$ then there are r pivot columns. The null space has dimension $n - r$.

3. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A.

a) How many rows and columns does A have?

b) If x is in $\mathbf{R}^n$, then how do you find $T(x)$?

c) In terms of A, how do you know if T is one-to-one? onto?

d) What is the range of T?

Solution.

a) A has m rows and n columns.

b) $T(x) = Ax$.

c) T is one-to-one if and only if A has a pivot in every column. T is onto if and only if A has a pivot in every row.

d) $\operatorname{Col} A$.

4. Let A be an invertible $n \times n$ matrix.

a) What can you say about the columns of A?

b) What are $\operatorname{rank}(A)$ and $\dim \operatorname{Nul} A$?

c) What do you know about $\det(A)$?

d) How many solutions are there to $Ax = b$? What are they?

e) What is $\operatorname{Nul} A$?

f) Do you know anything about the eigenvalues of A?

g) Do you know whether or not A is diagonalizable?

Solution.

a) The columns are linearly independent. They also span $\mathbf{R}^n$.

b) $\operatorname{rank}(A) = n$ and $\dim \operatorname{Nul} A = 0$.

c) $\det(A) \neq 0$.

d) The only solution is $x = A^{-1}b$.

e) $\operatorname{Nul} A = \{0\}$.

f) They are nonzero.

g) No, invertibility has nothing to do with diagonalizability.

5. Let A be an $n \times n$ matrix with characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.

a) What is the degree of $f(\lambda)$?

b) Counting multiplicities, how many (real and complex) eigenvalues does A have?

c) If $f(0) = 0$, what does this tell you about A?

d) How can you know if A is diagonalizable?

e) If $n = 3$ and A has a complex eigenvalue, how many real roots does $f(\lambda)$ have?

f) Suppose $f(c) = 0$ for some real number c. How do you find the vectors x for which $Ax = cx$?

g) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (real and complex) eigenvalues of A, counting multiplicities, then what is their sum? their product?

h) In general, do the roots of $f(\lambda)$ change when A is row reduced? Why or why not?

Solution.

a) n

b) n

c) A is not invertible, since 0 is an eigenvalue.

d) If f has n distinct roots, then A is diagonalizable. Otherwise, you have to check if the dimension of each eigenspace is equal to the algebraic multiplicity of the corresponding eigenvalue.

e) Complex roots come in pairs, so f has one real root.

f) You compute $\mathrm{Nul}(A - cI)$.

g) The sum is the trace of A, i.e. the sum of the diagonal entries of A. The product is the determinant of A.

h) Yes, row reduction does not preserve eigenvalues. For instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is row equivalent to $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

6. Describe Span $\left\{ \begin{pmatrix} -6 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right\}$.

Solution.

Let $W = \text{Span}\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{pmatrix} -6 \\ 7 \\ 2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \qquad \text{and} \qquad v_3 = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}.$$

First we compute a basis for W. Noting that W is the column space of

$$A = \begin{pmatrix} -6 & 3 & 4 \\ 7 & 2 & -1 \\ 2 & 4 & 2 \end{pmatrix},$$

we row reduce to obtain

$$\begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}.$$

The first two columns are the pivot columns, so a basis of W is $\{v_1, v_2\}$. This means that W is the plane in $\mathbf{R}^3$ spanned by v_1 and v_2.

Here's another way to think of W, which will actually give us an explicit equation describing W. First, let's find $W^\perp$. The question didn't ask us to do this, but it will help us anyway.

$$W^\perp = \text{Nul}(A^T) = \text{Nul}\begin{pmatrix} -6 & 7 & 2 \\ 3 & 2 & 4 \\ 4 & -1 & 2 \end{pmatrix}.$$

Row reducing A^T yields

$$\begin{pmatrix} 1 & 0 & 8/11 \\ 0 & 1 & 10/11 \\ 0 & 0 & 0 \end{pmatrix} \implies W^\perp = \text{Nul}(A^T) = \text{Span}\left\{ \begin{pmatrix} -8 \\ -10 \\ 11 \end{pmatrix} \right\}.$$

This means that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ is in } W \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -8 \\ -10 \\ 11 \end{pmatrix} = 0 \iff -8x - 10y + 11z = 0.$$

So W is the plane in $\mathbf{R}^3$ defined by $-8x - 10y + 11z = 0$.

7. Find a linear dependence relation among

$$v_1 = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 5 \\ 3 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 6 \end{pmatrix}, \quad v_4 = \begin{pmatrix} -1 \\ 4 \\ -5 \\ 1 \end{pmatrix}.$$

Which subsets of $\{v_1, v_2, v_3, v_4\}$ are linearly independent?

Solution.

A linear dependence relation is an equation of the form $c_1v_1+c_2v_2+c_3v_3+c_4v_4 = 0$, where c_1, c_2, c_3, c_4 are not all zero. This is the same as a nontrivial solution to the matrix equation

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 4 & 5 & -1 & 4 \\ 0 & 3 & 2 & -5 \\ 3 & -1 & 6 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0.$$

Row reducing the matrix yields

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = c_4 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hence every linear dependence relation has the form $c_4(-2v_1 + v_2 + v_3 + v_4) = 0$. Taking $c_4 = 1$, a linear dependence relation is $-2v_1 + v_2 + v_3 + v_4 = 0$.

Any proper subset of $\{v_1, v_2, v_3, v_4\}$ is linearly independent. If, for example, the set $\{v_1, v_2, v_3\}$ were linearly dependent, then there would exist a linear dependence relation $c_1v_1 + c_2v_2 + c_3v_3 = 0$. But this gives a linear dependence relation $c_1v_1 + c_2v_2 + c_3v_3 + 0v_4 = 0$, and we found above that no such relation exists (all four coefficients must be nonzero in any linear dependence relation).

8. Find the eigenvalues and bases for the eigenspaces of the following matrices. Diagonalize if possible.

$$\textbf{a)}\ A = \begin{pmatrix} 4 & -3 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 2 \end{pmatrix} \qquad \textbf{b)}\ A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

Solution.

a) This is an upper-triangular matrix, so the eigenvalues are 4, −2, and 2. Computing the null spaces of $A-4I$, $A+2I$, and $A-2I$ yields bases

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right\}, \qquad \text{and} \qquad \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

for the 4-, (−2)-, and 2-eigenspaces, respectively. Therefore,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}.$$

b) We compute the characteristic polynomial

$$f(\lambda) = -\lambda^3 + 12\lambda + 16.$$

We guess that f has an integer root, which would then have to divide 16. This works, and we factor:

$$f(\lambda) = -(\lambda+2)^2(\lambda-4).$$

Computing the null spaces of $A+2I$ and $A-4I$ yields bases

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\} \qquad \text{and} \qquad \left\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right\}$$

of the (−2)- and 4-eigenspaces, respectively. Therefore,

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1}.$$

9. Find the least squares solution of the system of equations

$$\begin{aligned} x + 2y \phantom{{}+z} &= 0 \\ 2x + y + z &= 1 \\ 2y + z &= 3 \\ x + y + z &= 0 \\ 3x + 2z &= -1. \end{aligned}$$

Solution.

We have to find the least squares solution to $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix}.$$

We compute:

$$A^T A = \begin{pmatrix} 1 & 2 & 0 & 1 & 3 \\ 2 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 15 & 5 & 9 \\ 5 & 10 & 4 \\ 9 & 4 & 7 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 15 & 5 & 9 & -1 \\ 5 & 10 & 4 & 7 \\ 9 & 4 & 7 & 2 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{187}{185} \\ 0 & 1 & 0 & \frac{137}{185} \\ 0 & 0 & 1 & \frac{43}{37} \end{array}\right).$$

Hence the least squares solution is

$$\widehat{x} = \left(-\frac{187}{185}, \frac{137}{185}, \frac{43}{37}\right).$$

10. Find A^{10} if $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

Solution.

This is a diagonalization problem. The characteristic polynomial is

$$f(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4.$$

We guess that f has an integer root, which then must divide 4. This works, and we factor

$$f(\lambda) = -(\lambda - 1)(\lambda - 2)^2.$$

We compute bases for the 1- and 2-eigenspaces, respectively, to be

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

It follows that

$$A = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$

and therefore,

$$\begin{aligned} A^{10} &= \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 2^{10} \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1022 & 0 & -2046 \\ 1023 & 1024 & 1023 \\ 0123 & 0 & 2047 \end{pmatrix}. \end{aligned}$$

11. Let $V = \mathrm{Span}\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \qquad v_2 = \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \qquad v_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}.$$

a) Find an orthogonal basis for V.

b) Compute the matrix for the orthogonal projection proj_V.

Solution.

a) We run Gram–Schmidt on $\{v_1, v_2, v_3\}$:

$$\begin{aligned}
u_1 &= v_1 &&= \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\\
u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1}u_1 &= v_2 - \frac{3}{4}u_1 &&= \frac{1}{4}\begin{pmatrix}-3\\1\\1\\1\end{pmatrix}\\
u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1}u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2}u_2 &= v_3 - \frac{1}{2}u_1 - \frac{2}{3}u_2 &&= \frac{1}{3}\begin{pmatrix}0\\-2\\1\\1\end{pmatrix}.
\end{aligned}$$

Hence an orthogonal basis for V is

$$\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \frac{1}{4}\begin{pmatrix}-3\\1\\1\\1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}0\\-2\\1\\1\end{pmatrix}\right\}.$$

b) We compute the columns of the orthogonal projection:

$$\begin{aligned}
\mathrm{proj}_V(e_1) &= \frac{e_1\cdot u_1}{u_1\cdot u_1}u_1 + \frac{e_1\cdot u_2}{u_2\cdot u_2}u_2 + \frac{e_1\cdot u_3}{u_3\cdot u_3}u_3 \\
&= \frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{-3/4}{3/4}\frac{1}{4}\begin{pmatrix}-3\\1\\1\\1\end{pmatrix} + \frac{0}{2/3}\frac{1}{3}\begin{pmatrix}0\\-2\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \\
\mathrm{proj}_V(e_2) &= \frac{e_2\cdot u_1}{u_1\cdot u_1}u_1 + \frac{e_2\cdot u_2}{u_2\cdot u_2}u_2 + \frac{e_2\cdot u_3}{u_3\cdot u_3}u_3 \\
&= \frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{1/4}{3/4}\frac{1}{4}\begin{pmatrix}-3\\1\\1\\1\end{pmatrix} + \frac{-2/3}{2/3}\frac{1}{3}\begin{pmatrix}0\\-2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\
\mathrm{proj}_V(e_3) &= \frac{e_3\cdot u_1}{u_1\cdot u_1}u_1 + \frac{e_3\cdot u_2}{u_2\cdot u_2}u_2 + \frac{e_3\cdot u_3}{u_3\cdot u_3}u_3 \\
&= \frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{1/4}{3/4}\frac{1}{4}\begin{pmatrix}-3\\1\\1\\1\end{pmatrix} + \frac{1/3}{2/3}\frac{1}{3}\begin{pmatrix}0\\-2\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} \\
\mathrm{proj}_V(e_4) &= \frac{e_4\cdot u_1}{u_1\cdot u_1}u_1 + \frac{e_4\cdot u_2}{u_2\cdot u_2}u_2 + \frac{e_4\cdot u_3}{u_3\cdot u_3}u_3 \\
&= \frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{1/4}{3/4}\frac{1}{4}\begin{pmatrix}-3\\1\\1\\1\end{pmatrix} + \frac{1/3}{2/3}\frac{1}{3}\begin{pmatrix}0\\-2\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}
\end{aligned}$$

Hence the matrix for proj_V is

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1/2&1/2\\0&0&1/2&1/2\end{pmatrix}.$$

12. Find the determinant of the matrix

$$A = \begin{pmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{pmatrix}.$$

Solution.

This is a big, complicated matrix, so it's probably best to find the determinant using row reduction. The result is $\det(A) = 585$.

13. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{pmatrix}.$$

a) Find a basis for Col$A$.

b) Describe Col$A$ geometrically.

c) Find a basis for Nul$A$.

d) Describe Nul$A$ geometrically.

Solution.

First we row reduce A to get

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) The only pivot column is the first, so a basis for Col$A$ is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}.$$

b) This is the line through the first column of A.

c) The parametric vector form of $Ax = 0$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix},$$

which is the plane with basis

$$\left\{ \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

d) As in Problem 6, we can compute that this is the plane in $\mathbf{R}^3$ defined by the equation $x + 4y + 2z = 0$.

14. Find numbers a, b, c, and d such that the linear system corresponding to the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{array}\right)$$

has **a)** no solutions, and **b)** infinitely many solutions.

Solution.

a) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ has no solutions.

b) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ has infinitely many solutions.

15. Celia has one hour to spend at the CRC, and she wants to jog, play handball, and ride a stationary bike. Jogging burns 13 calories per minute, handball burns 11, and cycling burns 7. She jogs twice as long as she rides the bike. How long should she participate in each of these activities in order to burn exactly 660 calories?

Solution.

Let x be the number of minutes spent jogging, y the number of minutes playing handball, and z the number of minutes cycling. The conditions of the problem require

$$\begin{aligned} x + y + z &= 60 \\ 13x + 11y + 7z &= 660 \\ x \qquad - 2z &= 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{pmatrix} 1 & 1 & 1 \\ 13 & 11 & 7 \\ 1 & 0 & -2 \end{pmatrix} x = \begin{pmatrix} 60 \\ 660 \\ 0 \end{pmatrix}.$$

We solve the matrix equation by using an augmented matrix and row reducing:

$$\begin{pmatrix} 1 & 1 & 1 & 60 \\ 13 & 11 & 7 & 660 \\ 1 & 0 & -2 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 60 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 60 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}.$$

So Celia should spend $2z$ minutes jogging, $60 - 3z$ minutes playing handball, and z minutes cycling, for any value of z strictly between 0 and 20 minutes (since she wants to do all three).

16. Consider the matrix

$$A = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}.$$

a) Find the complex eigenvalues and eigenvectors of A.

b) Find an invertible matrix P and a rotation-scaling matrix C such that $A = PCP^{-1}$.

c) By how much does C rotate and scale?

d) Describe how repeated multiplication by A acts on the plane.

Solution.

a) The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 8\lambda + 17.$$

Using the quadratic formula, we find

$$\lambda = \frac{8 \pm \sqrt{64 - 4 \cdot 17}}{2} = 4 \pm i.$$

These are the eigenvalues. To find an eigenvector with eigenvalue $\lambda = 4 - i$, we compute

$$A - \lambda I = \begin{pmatrix} 1+i & -2 \\ \star & \star \end{pmatrix} \xrightarrow{\text{eigenvector}} v = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

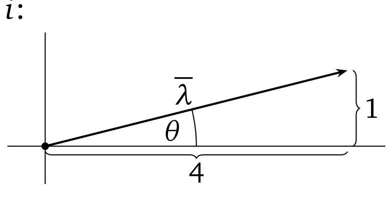
(the second row is necessarily a multiple of the first). Hence an eigenvector with eigenvalue $\overline{\lambda} = 4 + i$ is

$$\overline{v} = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}.$$

b) Taking $\lambda = 4 - i$ and v as above, we can use

$$P = \begin{pmatrix} \mathrm{Re}(v) & \mathrm{Im}(v) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix}.$$

c) The scaling factor is $|\lambda| = \sqrt{4^2 + 1^2} = \sqrt{17}$. The rotation is by the argument θ of $\overline{\lambda} = 4 + i$:



$$\theta = \tan^{-1}(1/4) \approx .2450$$

d) Multiplying by A repeatedly causes a point to spiral outwards, counterclockwise around the ellipse centered at the origin and through the points $(2, 1)$ and $(0, 1)$.