Name:______________________________ Recitation Section:____________

Math 1553 Lecture A, Quiz 4: Linear transformations (10 points, 10 minutes)

Solutions

Show your work and justify answers where appropriate. If you write the correct answer without sufficient work or justification, you will receive little or no credit.

1. (1 point each) True or False. Circle TRUE if the statement is *always* true. Otherwise, circle FALSE. You do not need to justify your answer.

a) If $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation, then T is one-to-one.
TRUE $\boxed{\text{FALSE}}$ (for example $T(x,y) = (x,0,0)$)

b) If T is the linear transformation whose standard matrix is $\begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 5 \end{pmatrix}$, then the domain of T is $\mathbf{R}^3$.
$\boxed{\text{TRUE}}$ FALSE

2. (3 points) Write the standard matrix A for the transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that first reflects about the line $y = x$, then rotates counterclockwise by $\frac{\pi}{2}$ radians.
Show your work!

Solution.

$A = \begin{pmatrix} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$.

Reflection about $y = x$ sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and then the rotation sends $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, so $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

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Thus $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Alternatively, we recently learned matrix multiplication. We put the reflection matrix on the right and the rotation matrix on the left:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Turn over to the back side for problem 3!)

3. (5 points) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x-2z \\ 2y \end{pmatrix}$.

a) Write the standard matrix A for T.

b) Is T onto? If your answer is yes, justify why. If your answer is no, find a vector $\mathbf{b}$ in $\mathbf{R}^3$ which is not in the range of T.

Solution.

a) $A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$.

b) No. We could row-reduce to find that A only has two pivots (thus does not have a pivot in every row). Alternatively, we could skip that justification by simply finding a vector which is not in the range of T.

Method 1: From the formula for T, we see that any vector $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in the range of T satisfies $b_3 = 2b_1$, so for example $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is not in the range of T.

Method 2: If $T(x) = b$ then the system $\left(\begin{array}{ccc|c} 0 & 1 & 0 & b_1 \\ 1 & 0 & -2 & b_2 \\ 0 & 2 & 0 & b_3 \end{array}\right)$ is consistent, and subtracting twice the first row from the third shows $b_3 = 2b_1$ (otherwise the final column would have a pivot) so any vector satisfying $b_3 \neq 2b_1$ is not in the range of T, for example $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.