Math 1553: Some Additional Final Exam Practice Problems

Spring 2022

These problems are for extra practice for the final. They are not meant to be 100% comprehensive in scope.

1. Define the following terms: span, linear combination, linearly independent, linear transformation, column space, null space, transpose, inverse, dimension, rank, eigenvalue, eigenvector, eigenspace, diagonalizable, orthogonal.

2. Let A be an $m \times n$ matrix.

a) How do you determine the pivot columns of A?

b) What do the pivot columns tell you about the equation $Ax = b$?

c) What space is equal to the span of the pivot columns?

d) What is the difference between solving $Ax = b$ and $Ax = 0$? How are the two solution sets related geometrically?

e) If $\operatorname{rank}(A) = r$, where $0 \leq r \leq n$, then how many columns have pivots? What is the dimension of the null space?

3. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A.

a) How many rows and columns does A have?

b) If x is in $\mathbf{R}^n$, then how do you find $T(x)$?

c) In terms of A, how do you know if T is one-to-one? onto?

d) What is the range of T?

4. Let A be an invertible $n \times n$ matrix.

a) What can you say about the columns of A?

b) What are $\operatorname{rank}(A)$ and $\dim \operatorname{Nul} A$?

c) What do you know about $\det(A)$?

d) How many solutions are there to $Ax = b$? What are they?

e) What is $\operatorname{Nul} A$?

f) Do you know anything about the eigenvalues of A?

g) Do you know whether or not A is diagonalizable?

5. Let A be an $n \times n$ matrix with characteristic polynomial $f(\lambda) = \det(A - \lambda I)$. (note: your instructor may have defined the characteristic polynomial as $\det(\lambda I - A)$. In either case, A will have the same eigenvalues and eigenvectors)

a) What is the degree of $f(\lambda)$?

b) Counting multiplicities, how many (real and complex) eigenvalues does A have?

c) If $f(0) = 0$, what does this tell you about A?

d) How can you know if A is diagonalizable?

e) If $n = 3$ and A has a complex eigenvalue, how many real roots does $f(\lambda)$ have?

f) Suppose $f(c) = 0$ for some real number c. How do you find the vectors x for which $Ax = cx$?

g) In general, do the roots of $f(\lambda)$ change when A is row reduced? Why or why not?

6. Find numbers a, b, c, and d such that the linear system corresponding to the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{array}\right)$$

has **a)** no solutions, and **b)** infinitely many solutions.

7. Celia has one hour to spend at the CRC, and she wants to jog, play handball, and ride a stationary bike. Jogging burns 13 calories per minute, handball burns 11, and cycling burns 7. She jogs twice as long as she rides the bike. How long should she participate in each of these activities in order to burn exactly 660 calories?

8. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that rotates counterclockwise by $\dfrac{\pi}{6}$ radians, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects about the line $y = x$.

a) Find the standard matrix A for T and the standard matrix B for U.

b) Find the matrix for T^{-1} and the matrix for U^{-1}. Clearly label your answers.

c) Compute the matrix M for the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ that first rotates *clockwise* by $\frac{\pi}{6}$ radians, then reflects about the line $y = x$, then rotates counterclockwise by $\frac{\pi}{6}$ radians.

9. Let $W = \operatorname{Span}\left\{ \begin{pmatrix} -6 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right\}$. Find a basis for W and a basis for $W^{\perp}$.

10. Find a linear dependence relation among

$$v_1 = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 5 \\ 3 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 6 \end{pmatrix}, \quad v_4 = \begin{pmatrix} -1 \\ 4 \\ -5 \\ 1 \end{pmatrix}.$$

Which subsets of $\{v_1, v_2, v_3, v_4\}$ are linearly independent?

11. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{pmatrix}.$$

a) Find a basis for Col$A$.

b) Describe Col$A$ geometrically.

c) Find a basis for Nul$A$.

d) Describe Nul$A$ geometrically.

12. Find the determinant of the matrix

$$A = \begin{pmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{pmatrix}.$$

13. Let $A = \begin{pmatrix} 2 & -6 \\ 2 & 2 \end{pmatrix}$.
(a) Find the characteristic polynomial of A.
(b) Find the complex eigenvalues of A. Fully simplify your answer.
(c) For the eigenvalue with negative imaginary part, find a corresponding eigenvector.

14. Find the eigenvalues and bases for the eigenspaces of the following matrices. Diagonalize if possible.

$$\textbf{a)}\ A = \begin{pmatrix} 4 & -3 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 2 \end{pmatrix} \qquad \textbf{b)}\ A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

Hint: the eigenvalues in (b) are $\lambda = -2$ and $\lambda = 4$.

15. Find the least squares solution of the system of equations

$$\begin{aligned} x + 2y \phantom{{}+z} &= 0 \\ 2x + y + z &= 1 \\ 2y + z &= 3 \\ x + y + z &= 0 \\ 3x \phantom{{}+y} + 2z &= -1. \end{aligned}$$

16. Find A^{10} if $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

17. Let $V = \mathrm{Span}\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

a) Find a basis for V.

b) Compute the matrix for the orthogonal projection onto V.

18. Let W be the set of all vectors in $\mathbf{R}^3$ of the form $(x, x-y, y)$ where x and y are real numbers.

a) Find a basis for $W^\perp$.

b) Find the matrix B for orthogonal projection onto W.

c) Diagonalize B by finding an invertible matrix C and diagonal matrix D so that $B = CDC^{-1}$.

19. Find, and draw, the best fit line $y = Mx + B$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$. Set up an equation (but do not solve) for the best-fit parabola $y = Ax^2 + Bx + C$ through those same data points.

Even more practice problems

Here is an additional list of practice problems.

Note: Answers to the remaining problems will **not** be posted.

0. Write down (and understand!!) the definitions of:

- Linear Dependence and Independence of Vectors
- Span of Vectors
- Echelon Form and Reduced Echelon Form
- Basis
- Subspace
- NullSpace of a Matrix
- Invertible and Non-invertible Matrix (Non-singular and singular matrix).
- Rank of a Matrix
- Column Space of a Matrix
- Row Space of a Matrix
- Dimension of Subspace
- Determinant of a Matrix
- Eigenvalue of a Matrix
- Eigenvector of a Matrix
- Characteristic polynomial of a Matrix
- Eigenspace corresponding to an Eigenvalue
- Algebraic and Geometric Multiplicity of an Eigenvalue
- Diagonalizable Matrix
- Dot Product
- Orthogonal Vectors
- Orthogonal Complement
- Orthogonal Projection
- Least Squares Solution

1. Suppose that A is a 3×3 matrix with eigenvalues 1, 2 and -5.

a) Find the determinant of A.

b) Is the matrix A invertible?

c) Is the matrix A diagonalizable?

d) Find the characteristic polynomial of A.

e) Find the eigenvalues of A^5.

f) Find the eigenvalues of BAB^{-1} where B is any invertible matrix.

2. V is a subspace of $\mathbf{R}^4$ spanned by the vectors $\begin{pmatrix} 2 \\ 3 \\ -1 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 7 \\ -1 \\ 2 \end{pmatrix}$. Let A be the 4×4 matrix with columns given by the 4 vectors above.

a) Find a basis of V, the dimension of V, the dimension of the kernel of A and the rank of A.

b) Is the matrix A invertible? Explain.

d) Is the vector $\begin{pmatrix} 0 \\ 1 \\ -3 \\ 4 \end{pmatrix}$ in V? If not, find the closest vector in V to it.

e) Find a basis of the null space of A.

f) Without performing further calculations, find determinant of A and explain your answer.

g) Find one eigenvalue and one eigenvector of A.

h) Repeat parts a)-g) with vectors $\begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ -2 \\ 4 \end{pmatrix}$.

3. T is a linear transformation with $T\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(a) Find $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) Find a vector $\mathbf{u}$ such that $T(\mathbf{u}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) Find the domain of T.

(d) Find the range of T.

(e) Find the matrix of T.

(f) Without doing further calculations, find the rank of the matrix of T and explain your answer.

(g) Does there exist a non-zero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$? Explain.

4. Consider the equation $A\mathbf{x} = \mathbf{b}$ with $A = \begin{pmatrix} 1 & -1 \\ 0 & -2 \\ 3 & 4 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 3 \\ 1 \\ 1 \end{pmatrix}$.

(a) Does this equation have a solution? If yes, find all solutions, if not, explain.

(b) Find the least squares solution of the above equation.

(c) Find the length and the dot product of the columns of A. Are the columns of A orthogonal?

(d) Find a pair of rows of A that are perpendicular to each other.

(e) Find the dimension of the kernel of A.

5. Find the eigenvalues, eigenvectors and diagonalize the following matrices, if possible.

(a) $\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$.

(b) $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

(d) $\begin{pmatrix} 5 & 3 & 1 \\ 0 & 5 & 1 \\ 1 & 0 & 2 \end{pmatrix}$.

6. Find the best fit line with equation $y = mx + b$ to the following sets of points:

(a) $(1,2)$, $(2,4)$, $(-1,0)$, $(5,2)$, $(3,3)$.

(b) $(2,-1)$, $(0,0)$, $(5,4)$, $(-1,2)$.

7. True or False. No partial credit.

(a) The span of the columns of a matrix A is equal to the range of the linear transformation T given by $T(\mathbf{x}) = A\mathbf{x}$.

(b) Any system of equations $A\mathbf{x} = \mathbf{b}$ has a least squares solution.

(c) Any 4 linearly independent vectors in $\mathbf{R}^4$ form a basis of $\mathbf{R}^4$.

(d) If the matrix A has more columns than rows then the system $A\mathbf{x} = \mathbf{0}$ always has infinitely many solutions.

(e) Any invertible matrix can be diagonalized.

(f) Any diagonalizable matrix is invertible.

(g) If $\mathbf{u}$ is perpendicular to every vector in the basis of a subspace V, then the orthogonal projection of $\mathbf{u}$ onto V is the zero vector.

(h) If the characteristic polynomial of A is $(\lambda-1)^2(\lambda-2)^2$ then the determinant of A is 2.

(i) For an invertible matrix A, the eigenvectors of A^{-1} are the same as eigenvectors of A.

(j) If a matrix A is not invertible then equation $A\mathbf{x}=\mathbf{b}$ has either no solutions of infinitely many solutions.

(k) If a matrix A is invertible then equation $A\mathbf{x}=\mathbf{b}$ always has a unique solution.

(l) If a $n\times n$ matrix A has linearly independent rows then A is invertible.

(m) If a $n\times n$ matrix A has linearly independent columns then A is invertible.

(n) If A is an invertible matrix then A^TA is also invertible.

(o) If 7×9 matrix A has kernel of dimension 5 then the column space of A has dimension 2.

(p) Any linearly independent set of vectors is a basis of its span.

(q) The eigenvalues of A are the same as eigenvalues of A^T.

(r) If a vector $\mathbf{u}$ is orthogonal to all rows of A then $\mathbf{u}$ is in the null space of A.

8. $A=\begin{pmatrix} 3 & s \\ 1 & -1 \end{pmatrix}$. Find a number s so that:

(a) A is non-invertible.

(b) A is not diagonalizable.

(c) 3 is an eigenvalue of A.

(d) Columns of A are orthogonal.

(e) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector of A.

(f) A^{-1} has eigenvalue 4.

(g) A^{-1} has eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

9. $A = \begin{pmatrix} 3 & -3 & 0 \\ 3 & -1 & 2 \\ b & 0 & 2 \end{pmatrix}$. Find a number b (if possible) so that:

(a) The determinant of A is 4.

(b) The rank of A is 2.

(c) $\frac{1}{2}$ is an eigenvalue of A^{-1}.

(d) The system $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has no solutions.

(e) The system $A\mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ has infinitely many solutions.

10. Find a 3×3 matrix with column space spanned by $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and null space spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.