Math 1553 Worksheet §2.3, S2.4

Solutions

1. True or false. If the statement is *always* true, answer True. Otherwise, answer False. In parts (a) and (b), A is an $m \times n$ matrix and b is a vector in $\mathbf{R}^m$.

a) If b is in the span of the columns of A, then the matrix equation $Ax = b$ is consistent.

b) If $Ax = b$ is inconsistent, then A does not have a pivot in every column.

c) If A is a 4×3 matrix, then the equation $Ax = b$ is inconsistent for some b in $\mathbf{R}^4$.

d) Suppose A is a 3×3 matrix with two pivots, and suppose that b is a vector so that $Ax = b$ is consistent. Then the solution set for $Ax = b$ is a plane.

Solution.

a) True. Let the columns of A be $v_1, \cdots, v_n$. Since b in $\mathrm{Span}\{v_1, \cdots, v_n\}$, this means b can be written as a linear combinations of these column vectors, so

$$x_1 v_1 + \cdots + x_n v_n = b$$

for some scalars $x_1, \ldots, x_n$. Therefore, $Ax = b$ where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

b) False, for instance consider

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This is an inconsistent system even though A has a pivot in each column.

c) True. Any 4×3 matrix A will have at most 3 pivots, so A cannot have a pivot in every row. For example, consider the augmented matrix $\left(A \mid b \right)$ below.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

d) False. The matrix A has two pivots, which means that the *column span* of A is plane, but this is not what the question is asking! It is asking about the solution set to $Ax = b$, not the column span of A.

Since $Ax = b$ corresponds to a system of 3 equations in 3 variables, the fact that A has two pivots means that the system will have exactly one free variable, so the solution set will be a line in $\mathbf{R}^3$.

2. Let

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}.$$

Solve the matrix equation $Ax = b$ and write your answer in parametric form.

Solution.

We translate the matrix equation into an augmented matrix, and row reduce it:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The right column is not a pivot column, so the system is consistent.

The RREF of the augmented matrix gives

$$x_1 = 2 - 5x_3 \qquad x_2 = 3 - 4x_3 \qquad x_3 = x_3 \ \ (x_3 \text{ is free}).$$

If we wanted to write just one specific solution, we could take $x_3 = 0$ and that would give us $x_1 = 2, x_2 = 3$:

$$b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}.$$

3. Find the set of solutions to $x_1-3x_2+5x_3=0$ and write your answer in parametric vector form. Next, find the set of solutions to $x_1-3x_2+5x_3=3$ and write the solutions in parametric vector form. How do the solution sets compare geometrically?

Solution.

The homogeneous system $x_1-3x_2+5x_3=0$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 0 \end{array}\right)$, which has two free variables x_2 and x_3.

$$x_1 = 3x_2 - 5x_3 \qquad x_2 = x_2 \text{ (free)} \qquad x_3 = x_3 \text{ (free)}.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2-5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{x_2\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

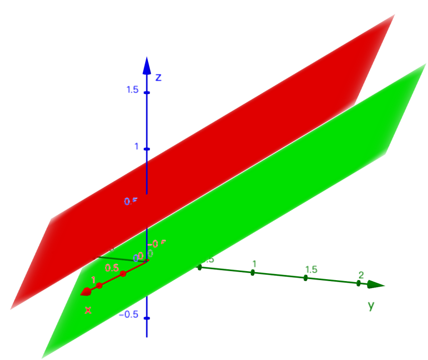
The solution set for $x_1-3x_2+5x_3=0$ is the plane spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.

The nonhomogeneous system $x_1-3x_2+5x_3=3$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 3 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3 + 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3+3x_2-5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

This solution set (red) is the *translation by* $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ of the plane (green) spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.



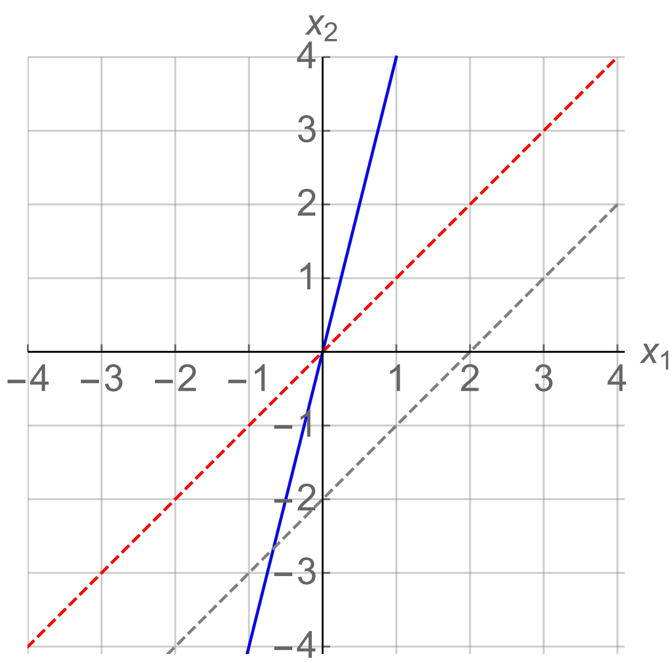
Here is the link to a 3D picture you can play with https://www.geogebra.org/calculator/j57ttsnb

4. Let $A = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}$. On the same graph, draw each of the following:
(a) The span of the columns of A.

(b) The set of solutions to $Ax = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(c) The set of solutions to $Ax = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$.
Label each of these clearly.

Solution.



The blue line is the span of columns of A: Span $\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$. If you draw the two column vectors, you will see they both fall on the line $x_2 = 4x_1$.

The red dashed line is the span of solutions of $Ax = 0$: Span $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. To see this is the case, you can row reduce the augmented matrix to RREF, which is $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right)$. That implies the solution set is the line $x_2 = x_1$.

The gray dashed line is the set of solutions to $Ax = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$. To see this is the case, you can row reduce the corresponding augmented matrix to RREF, which is $\left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array} \right)$. That implies the solution set is the line $x_1 = 2 + x_2$ (where x_2 is free) which yields parametric form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 + x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In other words, this solution set is the line through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ parallel to the span of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.