Supplemental problems: §2.5

1. Justify why each of the following true statements can be checked without row reduction.

a) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \right\}$ is linearly independent.

b) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$ is linearly independent.

c) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly dependent.

Solution.

a) You can eyeball linear independence: if

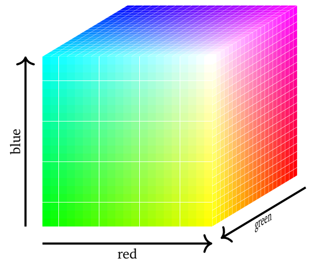
$$x \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix} + z \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 3x \\ 3x + y\sqrt{2} \\ 4x + \pi z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then $x = 0$, so $y = z = 0$ too.

b) Since the first coordinate of $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ is nonzero, $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ cannot be in the span of $\left\{ \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$. And $\begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}$ is not in the span of $\left\{ \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$ because it is not a multiple. Hence the span gets bigger each time you add a vector, so they're linearly independent.

c) Any four vectors in $\mathbf{R}^3$ are linearly dependent; you don't need row reduction for that.

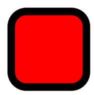
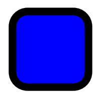
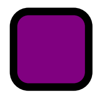
2. Every color on my computer monitor is a vector in $\mathbf{R}^3$ with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.



Given colors $v_1, v_2, \dots, v_p$, we can form a "weighted average" of these colors by making a linear combination

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

with $c_1 + c_2 + \cdots + c_p = 1$. Example:

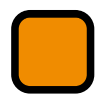
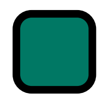
$\frac{1}{2}$  $+\ \frac{1}{2}$  $=$ 

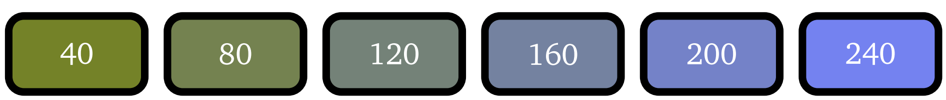
Consider the colors on the right. For which h is

$$\left\{ \begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}, \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}, \begin{pmatrix} 116 \\ 130 \\ h \end{pmatrix} \right\}$$

linearly dependent? What does that say about the corresponding color?

$$\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix} \quad \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}$$

$h =$ 

Solution.

The vectors

$$\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}, \quad \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}, \quad \begin{pmatrix} 116 \\ 130 \\ h \end{pmatrix}$$

are linearly dependent if and only if the vector equation

$$x \begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix} + y \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix} + z \begin{pmatrix} 116 \\ 130 \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a nonzero solution. This translates into the matrix

$$\begin{pmatrix} 180 & 100 & 116 \\ 50 & 150 & 130 \\ 200 & 100 & h \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & .2 \\ 0 & 1 & .8 \\ 0 & 0 & h-120 \end{pmatrix},$$

which has a free variable if and only if $\boxed{h = 120}$.

Suppose now that $h = 120$. The parametric form for the solution the above vector equation is

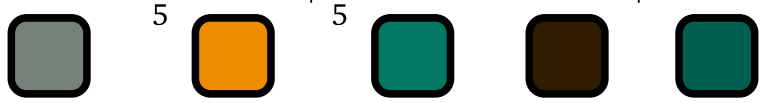
$$x = -.2z$$
$$y = -.8z.$$

Taking $z = 1$ gives the linear combination

$$-.2\begin{pmatrix}180\\50\\200\end{pmatrix} - .8\begin{pmatrix}100\\150\\100\end{pmatrix} + \begin{pmatrix}116\\130\\120\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

In terms of colors:

$$\begin{pmatrix}116\\130\\120\end{pmatrix} = \frac{1}{5}\begin{pmatrix}180\\50\\200\end{pmatrix} + \frac{4}{5}\begin{pmatrix}100\\150\\100\end{pmatrix} = \begin{pmatrix}36\\10\\40\end{pmatrix} + \begin{pmatrix}80\\120\\80\end{pmatrix}$$



3. Which of the following must be true for any set of seven vectors in $\mathbf{R}^5$? Answer "yes", "no", or "maybe" in each case.

a) The vectors span $\mathbf{R}^5$.

b) The vectors are linearly dependent.

c) At least one of the vectors is in the span of the other six vectors.

d) If we put the seven vectors as the columns of a matrix A, then the matrix equation $Ax = 0$ must have infinitely many solutions.

e) Suppose we put the seven vectors as the columns of a matrix A. Then for each b in $\mathbf{R}^5$, the matrix equation $Ax = b$ must be consistent.

f) If every vector b in $\mathbf{R}^5$ can be written as a linear combination of our seven vectors, then in fact every b in $\mathbf{R}^5$ can be written in *infinitely many* different ways as a linear combination of our seven vectors.

Solution.

a) Maybe.

b) Yes.

c) Yes.

d) Yes.

e) Maybe.

f) Yes. By assumption, the matrix A whose columns are our seven vectors has $\mathbf{R}^5$ as its column span, so A will have a pivot in every row. Therefore, A will have 5 pivot columns, so it will have 2 columns without pivots. This means that $Ax = b$ will be consistent no matter what b is, and there will be two free variables.

4. Suppose A is a 2×3 matrix and the solution set to $Ax = 0$ is Span$\left\{\begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$. Must it be true that the equation $Ax = b$ is consistent for each b in $\mathbf{R}^2$?

Solution.

Yes. The matrix equation $Ax = 0$ has one free variable because its solution set is a line. Since A is a 2×3 matrix, this means A has two pivots, so it has a pivot in every row. Therefore, the columns of A span $\mathbf{R}^2$, so the matrix equation $Ax = b$ is consistent for each b in $\mathbf{R}^2$.

5. Write vectors u, v, and w in $\mathbf{R}^4$ so that $\{u, v, w\}$ is linearly dependent, but u is not in Span$\{v, w\}$. **Solution.**

This is essentially a repeat of the 2.5 Webwork #8.

6. Suppose that A is a matrix with columns v_1, v_2, v_3, and v_4, where

$$v_1 - 2v_2 + 3v_3 + v_4 = 0.$$

Consider the set of vectors $\{v_1, v_2, v_3, v_4\}$.

a) Write one nonzero vector so that $Ax = 0$.

b) Must it be true that every vector in the set $\{v_1, v_2, v_3, v_4\}$ is a linear combination of the other vectors in the set? Justify your answer.

Solution.

a) By definition of multiplication,

$$A\begin{pmatrix}1\\-2\\3\\1\end{pmatrix} = v_1 - 2v_2 + 3v_3 + v_4 = 0.$$

b) Yes. Using the linear dependence relation we were given, we can write any vector in the set as a linear combination of the others.

$$\begin{aligned}
v_1 &= 2v_2 - 3v_3 - v_4,\\
v_2 &= \frac{1}{2}\left(v_1 + 3v_3 + v_4\right),\\
v_3 &= -\frac{1}{3}\left(v_1 - 2v_2 + v_4\right),\\
v_4 &= -v_1 + 2v_2 - 3v_3.
\end{aligned}$$

Supplemental problems: §§2.6, 2.7, 2.9

1. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

a) If A is a 3×100 matrix of rank 2, then $\dim(\mathrm{Nul} A) = 97$.

TRUE **FALSE**

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the columns of A form a basis for $\mathbf{R}^m$.

TRUE **FALSE**

c) The set $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - 4z = 0 \right\}$ is a subspace of $\mathbf{R}^4$.

TRUE **FALSE**

Solution.

a) False. By the Rank Theorem, $\mathrm{rank}(A) + \dim(\mathrm{Nul} A) = 100$, so $\dim(\mathrm{Nul} A) = 98$.

b) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its column space is a 2-dimensional subspace of $\mathbf{R}^3$.

c) True. V is $\mathrm{Nul}(A)$ for the 1×4 matrix A below, and therefore is automatically a subspace of $\mathbf{R}^4$:

$$A = \begin{pmatrix} 1 & 0 & -4 & 0 \end{pmatrix}.$$

Alternatively, we could verify the subspace properties directly if we wished, but this is much more work!

(1) The zero vector is in V, since $0 - 4(0)0 = 0$.

(2) Let $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$ be in V, so $x_1 - 4z_1 = 0$ and $x_2 - 4z_2 = 0$. We compute

$$u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}.$$

Is $(x_1 + x_2) - 4(z_1 + z_2) = 0$? Yes, since

$$(x_1 + x_2) - 4(z_1 + z_2) = (x_1 - 4z_1) \; + \; (x_2 - 4z_2) = 0 + 0 = 0.$$

(3) If $u = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ is in V then so is cu for any scalar c:

$$cu = \begin{pmatrix} cx \\ cy \\ cz \\ cw \end{pmatrix} \quad \text{and} \quad cx - 4cz = c(x - 4z) = c(0) = 0.$$

2. Write a matrix A so that $\operatorname{Col} A = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}$ and $\operatorname{Nul} A$ is the xz-plane.

Solution.

Many examples are possible. We'd like to design an A with the prescribed column span, so that $\left(A \mid 0\right)$ will have free variables x_1 and x_3. One way to do this is simply to leave the x_1 and x_3 columns blank, and make the second column $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$. This guarantees that A destroys the xz-plane and has the column span required.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

An alternative method for finding the same matrix: Write $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$. We want the column span to be the span of $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and we want

$$A\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = xv_1 + zv_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for all } x \text{ and } z.$$

One way to do this is choose $v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$.

3. Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) If $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace V of $\mathbf{R}^n$, then $\{v_1, v_2, v_3\}$ is a linearly independent set.

b) The solution set of a consistent matrix equation $Ax = b$ is a subspace.

c) A translate of a span is a subspace.

Solution.

a) True. If $\{v_1, v_2, v_3\}$ is linearly dependent then $\{v_1, v_2, v_3, v_4\}$ is automatically linearly dependent, which is impossible since $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace.

b) False. this is true if and only if $b = 0$, i.e., the equation is *homogeneous*, in which case the solution set is the null space of A.

c) False. A subspace must contain 0.

4. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) There exists a 3×5 matrix with rank 4.

b) If A is an 9×4 matrix with a pivot in each column, then

$$\mathrm{Nul} A = \{0\}.$$

c) There exists a 4×7 matrix A such that nullity $A = 5$.

d) If $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^4$, then $n = 4$.

Solution.

a) False. The rank is the dimension of the column space, which is a subspace of $\mathbf{R}^3$, hence has dimension at most 3.

b) True.

c) True. For instance,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

d) True. Any basis of $\mathbf{R}^4$ has 4 vectors.

5. Find bases for the column space and the null space of

$$A = \begin{pmatrix} 0 & 1 & -3 & 1 & 0 \\ 1 & -1 & 8 & -7 & 1 \\ -1 & -2 & 1 & 4 & -1 \end{pmatrix}.$$

Solution.

The RREF of $\left(A \mid 0\right)$ is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 5 & -6 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

so x_3, x_4, x_5 are free, and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -5x_3 + 6x_4 - x_5 \\ 3x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for Nul A is $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

To find a basis for Col A, we use the pivot columns as they were written in the *original* matrix A, not its RREF. These are the first two columns:

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}.$$

6. Find a basis for the subspace V of $\mathbf{R}^4$ given by

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x + 2y - 3z + w = 0 \right\}.$$

Solution.

V is Nul A for the 1×4 matrix $A = \begin{pmatrix} 1 & 2 & -3 & 1 \end{pmatrix}$. The augmented matrix $\left(A \mid 0 \right) = \left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 0 \end{array} \right)$ gives $x = -2y + 3z - w$ where y, z, w are free variables. The parametric vector form for the solution set to $Ax = 0$ is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2y + 3z - w \\ y \\ z \\ w \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for V is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

7. **a)** True or false: If A is an $m \times n$ matrix and $\text{Nul}(A) = \mathbf{R}^n$, then $\text{Col}(A) = \{0\}$.

b) Give an example of 2×2 matrix whose column space is the same as its null space.

c) True or false: For some m, we can find an $m \times 10$ matrix A whose column span has dimension 4 and whose solution set for $Ax = 0$ has dimension 5.

Solution.

a) If $\text{Nul}(A) = \mathbf{R}^n$ then $Ax = 0$ for all x in $\mathbf{R}^n$, so the only element in $\text{Col}(A)$ is $\{0\}$. Alternatively, the rank theorem says

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = n \implies \dim(\text{Col } A) + n = n \implies \dim(\text{Col } A) = 0 \implies \text{Col } A = \{0\}.$$

b) Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Its null space and column space are $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

c) False. The rank theorem says that the dimensions of the column space (Col$A$) and homogeneous solution space (Nul$A$) add to 10, no matter what m is.

8. Suppose V is a 3-dimensional subspace of $\mathbf{R}^5$ containing $\begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 9 \\ 8 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

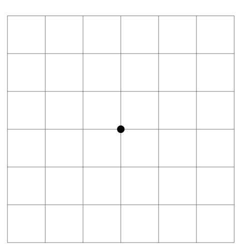
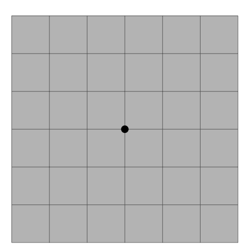
Is $\left\{ \begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ a basis for V? Justify your answer.

Solution.

Yes. The Basis Theorem says that since we know $\dim(V) = 3$, our three vectors will form a basis for V if and only if they are linearly independent.

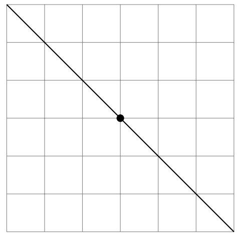
Call the vectors v_1, v_2, v_3. It is very little work to show that the matrix $A = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ has a pivot in every column, so the vectors are linearly independent.

9. **a)** Write a 2×2 matrix A with **rank** 2, and draw pictures of Nul$A$ and Col$A$.

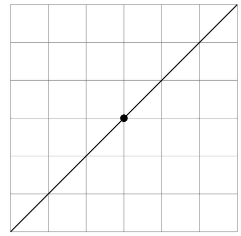
$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Nul A =  Col A = 

b) Write a 2×2 matrix B with **rank** 1, and draw pictures of Nul$B$ and Col$B$.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ Nul B =

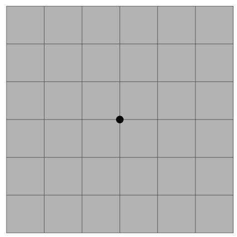


Col B =

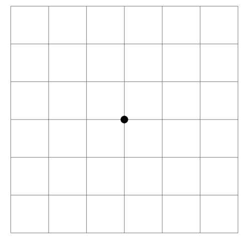


c) Write a 2×2 matrix C with **rank** 0, and draw pictures of Nul C and Col C.

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ Nul C =



Col C =



(In the grids, the dot is the origin.)

Supplemental problems: §3.1

1. Review from 2.6-2.9. Fill in the blanks: If A is a 7×6 matrix and the solution set for $Ax=0$ is a plane, then the column space of A is a ___4___-dimensional subspace of $\mathbf{R}^{\boxed{7}}$.

Reason: $\text{rank}(A)+\text{nullity}(A)=6 \qquad \text{rank}(A)+2=6 \qquad \text{rank}(A)=4$

2. Review from 2.6-2.9: Consider the matrix A below and its RREF:

$$A=\begin{pmatrix} 1 & 2 & -1 & -1 \\ -2 & -4 & -6 & 2 \\ 1 & 2 & -5 & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Write a basis for Col A.

The pivot columns (1 and 3) form a basis for Col(A), but really column 3 and any other column will work.

$$\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -6 \\ -5 \end{pmatrix}\right\}.$$

b) Find a basis for Nul A. From the RREF of A, we see the solution set is

$$x_1+2x_2-x_4=0, \quad x_3=0,$$

so $x_1=-2x_2+x_4$, x_2 and x_4 are free, and $x_3=0$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2+x_4 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} = x_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \qquad \text{A basis is} \quad \left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

c) Is there a matrix B so that Col(B) = Nul(A)? If yes, write such a B. If not, justify why no such matrix B exists.

Yes. Just take the columns of B to be a set whose span is Nul A, for example

$$B=\begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. Suppose T is a matrix transformation and the range of T is the subspace

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x - 3y + 4z = 0 \right\}$$

of $\mathbf{R}^3$, which contains the vectors $v_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$. Is $\{v_1, v_2\}$ a basis for the range of T?

Solution.

Yes. We know that V is a 2-dimensional subspace of $\mathbf{R}^3$ since $V = \mathrm{Nul}\begin{pmatrix} 1 & -3 & 4 \end{pmatrix}$ which corresponds to a homogeneous system with two free variables. Since $\{v_1, v_2\}$ is clearly a linearly independent set in V and $\dim(V) = 2$, it forms a basis for V by the Basis Theorem.

4. True or false. If the statement is *always* true, answer TRUE. Otherwise, circle FALSE.

a) The matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ performs reflection across the x-axis in $\mathbf{R}^2$. TRUE **[FALSE]** (T reflects across the y-axis then projects onto the x-axis)

b) The matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ performs rotation counterclockwise by $90°$ in $\mathbf{R}^2$. TRUE **[FALSE]** (T rotates clockwise $90°$)

5. Let T be the matrix transformation $T(x) = Ax$, where $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ -1 & 0 & -1 & -2 \\ 2 & 2 & 4 & 2 \end{pmatrix}$.

What is the domain of T? What is its codomain? Find a basis for the range of T and a basis for the kernel of T (the kernel of T is the set of all vectors satisfying $T(x) = 0$).

Solution: The domain of T is $\mathbf{R}^4$ and the codomain of T is $\mathbf{R}^3$. The range of T is Col A and the kernel of T is Nul A. We row-reduce $\begin{pmatrix} A \mid 0 \end{pmatrix}$:

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ -1 & 0 & -1 & -2 & 0 \\ 2 & 2 & 4 & 2 & 0 \end{array}\right) \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 + R_1} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 = R_1 - R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We see x_3 and x_4 are free, and $x_1 = -x_3 - 2x_4$ and $x_2 = -x_3 + x_4$. The parametric vector form for elements of Nul A is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 - 2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

A basis for kernel(T) is $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

A basis for Range(T) is given by the pivot columns of A, namely $\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$.

In this case, any two columns of A will actually form a basis for Col$A$, so any two columns of A will be a correct answer.

6. The matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ has RREF $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$. Define a matrix transformation by $T(x) = Ax$. Is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ a basis for the range of T?

Solution.

No. The range of T is Col A. To get a basis for Col A, we use the The pivot columns of the original matrix A, not its RREF.

7. In each case, a matrix is given below.

Match each matrix to its corresponding transformation (choosing from (i) through (viii)) by writing that roman numeral next to the matrix. Note there are four matrices and eight options, so not every option is used.

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ This is (i) Reflection across x-axis

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ This is (v) Rotation counterclockwise by $\pi/2$ radians

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ This is (viii) Projection onto the y-axis

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ This is (iii) Scaling by a factor of 2

(i) Reflection across x-axis
(ii) Reflection across y-axis
(iii) Scaling by a factor of 2
(iv) Scaling by a factor of 1/2
(v) Rotation counterclockwise by $\pi/2$ radians
(vi) Rotation clockwise by $\pi/2$ radians
(vii) Projection onto the x-axis
(viii) Projection onto the y-axis