MATH 1553, EXAM 3 SOLUTIONS
FALL 2023

Name		**GT ID**	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A, 8:25-9:15 AM) Kafer (B, 8:25-9:15 AM) Irvine (C, 9:30-10:20)

Kafer (D, 9:30-10:20 AM) He (G, 12:30-1:20 PM) Goldsztein (H, 12:30-1:20)

Goldsztein (I, 2:00-2:50 PM) Neto (L, 3:30-4:20 PM)

Yu (M, 3:30-4:20 PM) Ostrovskii, (N, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- Unless stated otherwise, the entries in all matrices are **real** numbers.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, November 15.

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Problem 1.

For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

In each statement, A is a matrix whose entries are **real numbers**.
Solutions are on the next page.

a) Suppose A is a 3×3 matrix and there is some b in $\mathbf{R}^3$ so that the equation $Ax = b$ has exactly one solution. Then A must be invertible.

[TRUE] FALSE

b) If A is an $n \times n$ matrix and $\det(A) = 0$, then $\lambda = 0$ must be an eigenvalue of A.

[TRUE] FALSE

c) There is a 3×3 real matrix A whose eigenvalues are -1, 3, and $2 + i$.

TRUE [FALSE]

d) Suppose A is a 4×4 matrix and its eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = 3, \quad \lambda_3 = 5, \quad \lambda_4 = 7.$$

Then A must be diagonalizable.

[TRUE] FALSE

e) If A is a 5×5 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(\lambda + 2)(\lambda - 4)^3,$$

then the null space of A must be a line.

[TRUE] FALSE

Solution.

a) True: If $Ax = b$ has exactly one solution for some b, then $Ax = 0$ must have exactly one solution, thus A is invertible by the Invertible Matrix Theorem (or by a direct argument using pivots).

b) True: If $\det(A) = 0$ then A is not invertible, which means $Ax = 0x$ has infinitely many solutions and therefore 0 is an eigenvalue of A. Alternatively, if $\det(A) = 0$ then A is not invertible so $A - 0I$ is not invertible, thus $\lambda = 0$ is an eigenvalue of A. This problem was inspired by a true/false question in the 5.1 Webwork #7.

c) False. This was basically taken from #4 of the 5.5 Webwork. If $2+i$ is an eigenvalue, then $2-i$ would also be an eigenvalue, whereby A would have 4 different eigenvalues (-1, 3, $2+i$, and $2-i$) which is impossible for a 3×3 matrix.

d) True. This is a quintessential diagonalization question. We know that eigenvectors for different eigenvalues are linearly independent. Since A has 4 different eigenvalues, this means we get 4 linearly independent eigenvectors in $\mathbf{R}^4$, therefore A is diagonalizable by the Diagonalization Theorem.

e) True. This is nearly identical to part of #3c in Sample Midterm 3A, but we give a full explanation below anyway. The eigenvalue $\lambda = 0$ has algebraic multiplicity 1 in the characteristic polynomial. Since for any eigenvalue we know

$$(\text{alg. mult.}) \geq (\text{geo. mult.}) \geq 1,$$

this gives us $1 \geq \text{geo. mult.} \geq 1$ for $\lambda = 0$, thus $\lambda = 0$ has geometric multiplicity 1. In other words, the null space (i.e. the 0-eigenspace) is a line.

Problem 2.

Solutions are on the next page.

a) (2 points) Let $A = \begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$. Find A^{-1}. Clearly circle your answer below.

[(i)] $\frac{1}{10}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ (ii) $\frac{1}{10}\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$ (iii) $-\frac{1}{2}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$

(iv) $\frac{1}{10}\begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$ (v) $\frac{1}{10}\begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$ (vi) $-\frac{1}{2}\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$

b) (3 points) Suppose A is an invertible matrix whose inverse is given by

$$A^{-1} = \begin{pmatrix} -1 & 2 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

(i) Suppose b is a vector in $\mathbf{R}^3$. How many solutions will the equation $Ax = b$ have? Circle your answer below.

None [Exactly one] Infinitely many solutions Not enough info to tell

(ii) Which **one** of the vectors below is a solution to $Ax = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$? Circle your answer.

$x = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ [$x = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$] $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $x = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ $x = \begin{pmatrix} 1 \\ 0 \\ -1/3 \end{pmatrix}$

c) (2 pts) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} 4a-2g & 4b-2h & 4c-2i \\ d & e & f \\ a & b & c \end{pmatrix}$.

Clearly circle your answer below.

(i) 1 (ii) -1 [(iii) 2] (iv) -2

(v) 4 (vi) -4 (vii) 8 (viii) -8.

d) (3 points) Suppose A and B are 2×2 matrices satisfying

$$\det(A) = 6, \qquad \det(B) = -3.$$

Which of the following statements must be true? Clearly circle all that apply.

[(i)] AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(ii) $\det(3B^{-1}) = -1$.

(iii) $A - 6I$ is not invertible.

Solution.

a) This is #2b from Sample Midterm 3 with changed numbers. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Here,

$$A^{-1} = \frac{1}{4(1)-(2)(-3)}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}.$$

b) A is invertible so $Ax = b$ is guaranteed to have exactly one solution. To solve $Ax = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ we have

$$x = A^{-1}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}.$$

c) This same type of problem has been on a worksheet, a Webwork, a quiz, and #2a on Sample Midterm 3A, so here on the exam we are seeing it for the fifth time. We get from $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ to $\begin{pmatrix} 4a-2g & 4b-2h & 4c-2i \\ d & e & f \\ a & b & c \end{pmatrix}$, we first swap rows 1 and 3 to get

$$\begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}. \quad \text{The new determinant is } 1(-1) = -1.$$

Next, we multiply the new first row by -2 to get

$$\begin{pmatrix} -2g & -2h & -2i \\ d & e & f \\ a & b & c \end{pmatrix}. \quad \text{The new determinant is } (-1)(-2) = 2.$$

Finally, we do a row-replacement that doesn't change the determinant to get

$$\begin{pmatrix} 4a-2g & 4b-2h & 4c-2i \\ d & e & f \\ a & b & c \end{pmatrix}. \quad \text{Determinant is still 2.}$$

d) (i) is true: $\det(A) \neq 0$ and $\det(B) \neq 0$, so both A and B are invertible and we know the classic formula $(AB)^{-1} = B^{-1}A^{-1}$.
(ii) is false because $\det(3B^{-1}) = 3^2\det(B^{-1}) = 3^2(-1/3) = -3$.
(iii) is false in general, for example $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ satisfies $\det(A) = 6$, however $A - 6I = \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix}$ which is certainly invertible.

Problem 3.

Solutions are on the next page.

a) Suppose A is an $n \times n$ matrix. Which **one** of the following statements is **not** correct?

(i) An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.

(ii) [An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.]

(iii) An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.

(iv) An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

b) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = 7x$, and let A be the standard matrix for T, so $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$.

In the blank below, write one eigenvector v in the (-1)-eigenspace of A.

$$\boxed{v = \begin{pmatrix} -7 \\ 1 \end{pmatrix}} \quad \text{or} \quad \boxed{v = \begin{pmatrix} 1 \\ -1/7 \end{pmatrix}}, \quad \text{etc.}$$

c) (2 points) Let $A = \begin{pmatrix} -1 & -4 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Which **one** of the following describes the 1-eigenspace of A?

(i) $\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$ (ii) Span $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ (iii) Span $\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$ (iv) Span $\left\{\begin{pmatrix} -1 \\ -4 \\ -6 \end{pmatrix}\right\}$

[(v) Span $\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}$] (vi) Span $\left\{\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}\right\}$ (vii) All of $\mathbf{R}^3$

d) (4 points) Let $A = \begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix}^{-1}$. Which of the following are true? Clearly circle all that apply.

[(i)] The eigenvalues of A are $1/2$ and 1.

(ii) For each vector x in $\mathbf{R}^2$, it is the case that $A^n x$ approaches $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ as n becomes large.

[(iii)] $\mathrm{Nul}(A - I) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$

[(iv)] $A^{10}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \left(\frac{1}{2}\right)^{10}\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

Solution.

a) This was copied and pasted from #2 in the 5.1-5.2 Supplement. The answer is (ii), because an eigenvalue is a scalar λ so that $(A-\lambda I)v = 0$ has a **non-trivial** solution.
If λ is not an eigenvalue of A, then it still satisfies $(A-\lambda I)v = 0$ for $v = 0$.

b) This problem is #2b from 5.1-5.2 Worksheet with a slightly modified line, and it was also in at least one of the sample midterms. Reflection across the line $y = 7x$ has eigenvalues 1 and -1.
The 1-eigenspace is the line $y = 7x$ itself. The (-1)-eigenspace is the line through $(0,0)$ perpendicular to $y = 7x$, which is the line $y = -(1/7)x$. Therefore, the (-1)-eigenspace is the span of $\begin{pmatrix} 1 \\ -1/7 \end{pmatrix}$ or equivalently the span of $\begin{pmatrix} -7 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$ etc.

c) $\left(A-I \mid 0\right) = \left(\begin{array}{ccc|c} -2 & -4 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so $x_1 + 2x_2 + 3x_3 = 0$ where x_2 and x_3 are free.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

d) A has been diagonalized for us, so its eigenvalues are $\lambda_1 = 1/2$ and $\lambda_2 = 1$, with corresponding eigenvectors $v_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, respectively. We use these facts below.
(i) True, the eigenvalues are 1/2 and 1.
(ii) False: in fact, this is not even true for $x = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

$$A^n \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \left(\frac{1}{2}\right)^n \begin{pmatrix} 4 \\ -2 \end{pmatrix} \text{ which approaches } \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ as } n \text{ gets very large.}$$

(iii) True: $\mathrm{Nul}(A-I)$ is by definition the 1-eigenspace of A, which we know is the span of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.
(iv) True: since $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ is a (1/2)-eigenvector, we have

$$A\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ -2 \end{pmatrix}, \qquad A^2\begin{pmatrix} 4 \\ -2 \end{pmatrix} = A\left(\frac{1}{2}\begin{pmatrix} 4 \\ -2 \end{pmatrix}\right) = \frac{1}{2}\cdot A\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \left(\frac{1}{2}\right)^2\begin{pmatrix} 4 \\ -2 \end{pmatrix}, \qquad \text{etc.,}$$

$$A^{10}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \left(\frac{1}{2}\right)^{10}\begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

Problem 4.

Solutions are on the next page.

a) (2 points) Find all values of c so that $\lambda = 2$ is an eigenvalue of the matrix $A = \begin{pmatrix} 4 & -3 \\ 4 & c \end{pmatrix}$. Clearly circle your answer below.

(i) $c = -3$ only [(ii) $c = -4$ only] (iii) $c = 4$ only (iv) $c = -6$ only

(v) All c except -4 (vi) All c except -6 (vii) All c except 6.

b) (4 points) The characteristic polynomial of some 7×7 matrix A is

$$\det(A - \lambda I) = (5 - \lambda)(1 - \lambda)^2(\pi - \lambda)^4.$$

For this particular matrix A, some information is given below.

Eigenvalue	5	1	π
Algebraic multiplicity	1	2	4
Geometric multiplicity	1	2	3

(i) In the table above, write its missing entries.

(ii) Is A diagonalizable? Clearly circle your answer below.

YES [NO] NOT ENOUGH INFORMATION

c) (4 points) Suppose A is a 3×3 matrix. Which of the following statements are true? Clearly circle all that apply.

(i) If B is a 3×3 matrix that has the same reduced row echelon form as A, then the eigenvalues of B are the same as the eigenvalues of A.

[(ii)] If $\lambda = 3$ is an eigenvalue of A, then the equation $Ax = 3x$ must have infinitely many solutions.

[(iii)] If the equation $(A - 2I)x = 0$ has only the trivial solution, then 2 is not an eigenvalue of A.

[(iv)] It is impossible for A to have 4 different eigenvalues.

Solution.

a) This is #7b of Sample Midterm 3A with the same eigenvalue and a slightly different matrix.

$\left(A-2I \mid 0\right) = \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 4 & c-2 & 0 \end{array}\right)$ which is non-invertible precisely when its determinant is 0, so

$$2(c-2)+12=0, \quad 2c=-8, \quad c=-4.$$

b) The eigenvalues and algebraic multiplicities come directly from the characteristic polynomial. The geometric multiplicity of $\lambda = 5$ must be 1 because its algebraic multiplicity is 1. The matrix A is NOT diagonalizable, because the geometric multiplicity of $\lambda = \pi$ is only 3 while its algebraic multiplicity is 4, leaving us with only 6 linearly independent eigenvectors in $\mathbf{R}^7$.

c) (i) is false: for example $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ have the same RREF but different eigenvalues.

(ii) is true directly from the definition of eigenvalue.

(iii) is true, because then $Ax = 2x$ has only the trivial solution.

(iv) is true: a 3×3 matrix has a degree 3 characteristic polynomial which has at most 3 roots, therefore A has at most 3 different eigenvalues.

Problem 5.

This entire problem was copied from the sample midterm, with a slightly new A.

For this problem, $A = \begin{pmatrix} -4 & -8 & 12 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{pmatrix}$.

a) (2 points) Write the eigenvalues of A. You do not need to show your work on this part.

This page is a quintessential diagonalization problem. For part (a), A is upper-triangular, so its eigenvalues are its diagonal entries: $\lambda = -4$ and $\lambda = -6$.

b) (5 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.

$$(A+4I|0) = \left(\begin{array}{ccc|c} 0 & -8 & 12 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & -8 & 12 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives x_1 free, $x_2 = 0$, and $x_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boxed{\text{Basis for (-4)-eigenspace : } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}}.$$

$$(A+6I|0) = \left(\begin{array}{ccc|c} 2 & -8 & 12 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 = R_1/2} \left(\begin{array}{ccc|c} 1 & -4 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives $x_1 - 4x_2 + 6x_3 = 0$ with x_2 free and x_3 free, so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_2 - 6x_3 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -6 \\ 0 \\ 1 \end{pmatrix} \qquad \boxed{\text{Basis for (-6)-eigenspace : } \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ 1 \end{pmatrix} \right\}}.$$

c) (3 points) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

We form C using linearly independent eigenvectors and form D using the eigenvalues written **in the corresponding order**. Many answers are possible. For example,

$$C = \begin{pmatrix} 1 & 4 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

or

$$C = \begin{pmatrix} 4 & -6 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Problem 6.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

a) Let $A = \begin{pmatrix} 5 & -5 \\ 4 & -3 \end{pmatrix}$.

(i) (4 points) Find the complex eigenvalues of A. Fully simplify your answer.

Solution:

$$\begin{aligned} 0 &= \det(A-\lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - (5-3)\lambda + (-15+20) \\ &= \lambda^2 - 2\lambda + 5, \end{aligned}$$

so

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(1)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = \boxed{1 \pm 2i}.$$

(ii) (3 points) For the eigenvalue with *positive* imaginary part, find one corresponding eigenvector v. Enter your answer in the space below.

Solution: $\left(A - (1+2i)I \mid 0\right)$ is

$$\left(\begin{array}{cc|c} 5-(1+2i) & -5 & 0 \\ 4 & -3-(1+2i) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 4-2i & -5 & 0 \\ 4 & -4-2i & 0 \end{array}\right) = \left(\begin{array}{cc|c} a & b & 0 \\ (*) & (*) & 0 \end{array}\right).$$

One eigenvector is $v = \begin{pmatrix} -b \\ a \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ 4-2i \end{pmatrix}}$ by the 2×2 eigenvector trick. Other answers possible, like $v = \begin{pmatrix} -5 \\ -4+2i \end{pmatrix}$ or $v = \begin{pmatrix} 4+2i \\ 4 \end{pmatrix}$ or even $v = \begin{pmatrix} \frac{2+i}{2} \\ 1 \end{pmatrix}$, etc.

b) (3 points) Given that

$$\det\begin{pmatrix} 0 & -1 & 3 \\ 0 & 4 & 2 \\ -2 & -1 & 1 \end{pmatrix} = 28, \quad \det\begin{pmatrix} 4 & 2 & -1 \\ 0 & 4 & 2 \\ -2 & -1 & 1 \end{pmatrix} = 8, \quad \text{and } \det\begin{pmatrix} 4 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 4 & 2 \end{pmatrix} = -56,$$

compute the determinant of the 4×4 matrix W below.

$$W = \begin{pmatrix} 4 & 1 & 2 & -1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 4 & 2 \\ -2 & -1 & -1 & 1 \end{pmatrix}$$

Solution: By cofactor expansion along the second column,

$$\begin{aligned} \det(W) &= 1C_{12} + 2C_{22} + 0C_{32} - 1C_{42} \\ &= 1(-1)^3(28) + 2(-1)^4(8) + 0 - 1(-1)^6(-56) \\ &= -28 + 16 + 56 = \boxed{44}. \end{aligned}$$

Problem 7.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) (5 points) Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & 4 \end{pmatrix}$. Find A^{-1}. Write your answer in the space below.

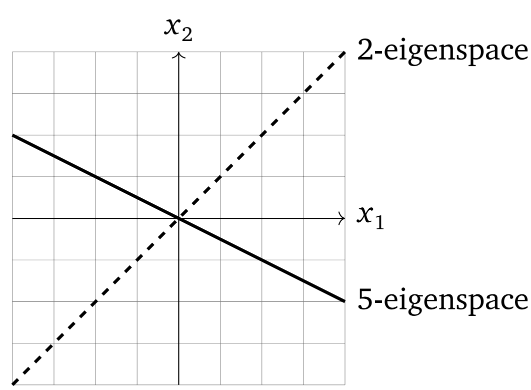
$$A^{-1} = \begin{pmatrix} 4 & 0 & -3 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$$

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 = -R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 = R_1 - 3R_3]{R_2 = R_2 + 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

b) (5 points) Let A be the 2×2 matrix whose 5-eigenspace is the **solid** line below and whose 2-eigenspace is the **dashed** line below. Find $A\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.



Solution: The 2-eigenspace is spanned by $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and the 5-eigenspace is spanned by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Also, note $\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, therefore

$$A\begin{pmatrix} 4 \\ 1 \end{pmatrix} = A\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) = A\begin{pmatrix} 2 \\ 2 \end{pmatrix} + A\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 2 \\ 2 \end{pmatrix} + 5\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} 10 \\ -5 \end{pmatrix} = \begin{pmatrix} 14 \\ -1 \end{pmatrix}.$$

Alternatively, we could have computed

$$\begin{aligned}
A\begin{pmatrix}4\\1\end{pmatrix} &= \begin{pmatrix}2 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 5\end{pmatrix}\begin{pmatrix}2 & 2\\2 & -1\end{pmatrix}^{-1}\begin{pmatrix}4\\1\end{pmatrix}\\
&= -\frac{1}{6}\begin{pmatrix}2 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 5\end{pmatrix}\begin{pmatrix}-1 & -2\\-2 & 2\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix}\\
&= -\frac{1}{6}\begin{pmatrix}2 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 5\end{pmatrix}\begin{pmatrix}-6\\-6\end{pmatrix}\\
&= -\frac{1}{6}\begin{pmatrix}2 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}-12\\-30\end{pmatrix}\\
&= -\frac{1}{6}\begin{pmatrix}-84\\6\end{pmatrix} = \begin{pmatrix}14\\-1\end{pmatrix}.
\end{aligned}$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.