

# Chapter 5

## Eigenvalues and Eigenvectors

# Section 5.1

## Eigenvalues and Eigenvectors

# A Biology Question

## Motivation

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have 0, 6, 8 baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$f_n$  = first-year rabbits in year  $n$

$s_n$  = second-year rabbits in year  $n$

$t_n$  = third-year rabbits in year  $n$

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let  $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$  and  $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$ . Then  $A v_n = v_{n+1}$ . ← difference equation

# A Biology Question

Continued

If you know  $v_0$ , what is  $v_{10}$ ?

$$v_{10} = Av_9 = AA v_8 = \dots = A^{10} v_0.$$

This makes it easy to compute examples by computer: [\[interactive\]](#)

$v_0$	$v_{10}$	$v_{11}$
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year:  $Av_n = v_{n+1} = 2v_n$ .
2. The ratios get close to (16 : 4 : 1):

$$v_n = (\text{scalar}) \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$

**Translation:** 2 is an eigenvalue, and  $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$  is an eigenvector!

## Definition

Let  $A$  be an  $n \times n$  matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of  $A$  is a *nonzero* vector  $v$  in  $\mathbf{R}^n$  such that  $Av = \lambda v$ , for some  $\lambda$  in  $\mathbf{R}$ . In other words,  $Av$  is a multiple of  $v$ .
2. An **eigenvalue** of  $A$  is a number  $\lambda$  in  $\mathbf{R}$  such that the equation  $Av = \lambda v$  has a *nontrivial* solution.

If  $Av = \lambda v$  for  $v \neq 0$ , we say  $\lambda$  is the **eigenvalue for**  $v$ , and  $v$  is an **eigenvector for**  $\lambda$ .

**Note:** Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

This is the most important definition in the course.

## Verifying Eigenvectors

### Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2v$$

Hence  $v$  is an eigenvector of  $A$ , with eigenvalue  $\lambda = 2$ .

### Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v$$

Hence  $v$  is an eigenvector of  $A$ , with eigenvalue  $\lambda = 4$ .

## Poll

Which of the vectors

A.  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  B.  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  C.  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$  D.  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  E.  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

are eigenvectors of the matrix  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ?

What are the eigenvalues?

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 2

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

eigenvector with eigenvalue 0

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 0

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

not an eigenvector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is never an eigenvector

## Verifying Eigenvalues

**Question:** Is  $\lambda = 3$  an eigenvalue of  $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$ ?

In other words, does  $Av = 3v$  have a nontrivial solution?

... does  $Av - 3v = 0$  have a nontrivial solution?

... does  $(A - 3I)v = 0$  have a nontrivial solution?

We know how to answer that! Row reduction!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Parametric form:  $x = -4y$ ; parametric vector form:  $\begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ .

Does there exist an eigenvector with eigenvalue  $\lambda = 3$ ? Yes! Any nonzero multiple of  $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ . Check:

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} -4 \\ 1 \end{pmatrix}. \quad \checkmark$$



## Definition

Let  $A$  be an  $n \times n$  matrix and let  $\lambda$  be an eigenvalue of  $A$ . The  $\lambda$ -**eigenspace** of  $A$  is the set of all eigenvectors of  $A$  with eigenvalue  $\lambda$ , plus the zero vector:

$$\begin{aligned}\lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\} \\ &= \text{Nul}(A - \lambda I).\end{aligned}$$

Since the  $\lambda$ -eigenspace is a null space, it is a *subspace* of  $\mathbf{R}^n$ .

How do you find a basis for the  $\lambda$ -eigenspace? Parametric vector form!

# Eigenspaces

## Example

Find a basis for the 3-eigenspace of

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}.$$

We have to solve the matrix equation  $A - 3I_2 = 0$ .

$$A - 3I_2 = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

$$\begin{array}{l} \text{RREF} \\ \rightsquigarrow \end{array} \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

$$\begin{array}{l} \text{parametric form} \\ \rightsquigarrow \end{array} x = -4y$$

$$\begin{array}{l} \text{parametric vector form} \\ \rightsquigarrow \end{array} \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\begin{array}{l} \text{basis} \\ \rightsquigarrow \end{array} \left\{ \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right\}.$$

# Eigenspaces

## Example

Find a basis for the 2-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

$$A - 2I = \begin{pmatrix} \begin{smallmatrix} 3 \\ 2 \\ 3 \end{smallmatrix} & 0 & 3 \\ - & 0 & -3 \\ - & 0 & -3 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{l} \text{parametric form} \\ \xrightarrow{\text{~~~~~}} \end{array} x = -2z$$

$$\begin{array}{l} \text{parametric vector form} \\ \xrightarrow{\text{~~~~~}} \end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{array}{l} \text{basis} \\ \xrightarrow{\text{~~~~~}} \end{array} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

# Eigenspaces

## Example

Find a basis for the  $\frac{1}{2}$ -eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

$$A - \frac{1}{2}I = \begin{pmatrix} 3 & 0 & 3 \\ -3/2 & 3/2 & -3 \\ -3/2 & 0 & -3/2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x = -z \\ y = z \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

# Eigenspaces

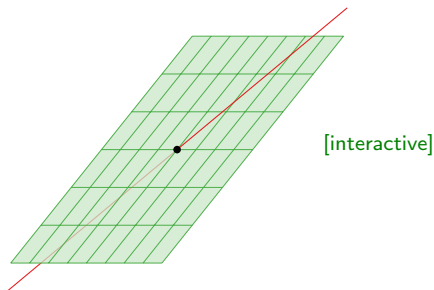
Example: picture

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

We computed bases for the 2-eigenspace and the  $1/2$ -eigenspace:

$$\text{2-eigenspace: } \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \frac{1}{2}\text{-eigenspace: } \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Hence the 2-eigenspace is a plane and the  $1/2$ -eigenspace is a line.



Let  $A$  be an  $n \times n$  matrix and let  $\lambda$  be a number.

1.  $\lambda$  is an eigenvalue of  $A$  if and only if  $(A - \lambda I)x = 0$  has a nontrivial solution, if and only if  $\text{Nul}(A - \lambda I) \neq \{0\}$ .
2. In this case, finding a basis for the  $\lambda$ -eigenspace of  $A$  means finding a basis for  $\text{Nul}(A - \lambda I)$  as usual, i.e. by finding the parametric vector form for the general solution to  $(A - \lambda I)x = 0$ .
3. The eigenvectors with eigenvalue  $\lambda$  are the nonzero elements of  $\text{Nul}(A - \lambda I)$ , i.e. the nontrivial solutions to  $(A - \lambda I)x = 0$ .

## The Eigenvalues of a Triangular Matrix are the Diagonal Entries

We've seen that finding eigenvectors for a given eigenvalue is a row reduction problem.

Finding all of the eigenvalues of a matrix *is not a row reduction problem!* We'll see how to do it in general next time. For now:

**Fact:** The eigenvalues of a triangular matrix are the diagonal entries.

**Why?**  $\text{Nul}(A - \lambda I) \neq \{0\}$  if and only if  $A - \lambda I$  is not invertible, if and only if  $\det(A - \lambda I) = 0$ .

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 0 & -1 & -2 & 7 \\ 0 & 0 & 8 & 12 \\ 0 & 0 & 0 & -3 \end{pmatrix} - \lambda I_4 = \begin{pmatrix} 3 - \lambda & 4 & 1 & 2 \\ 0 & -1 - \lambda & -2 & 7 \\ 0 & 0 & 8 - \lambda & 12 \\ 0 & 0 & 0 & -3 - \lambda \end{pmatrix}.$$

The determinant is  $(3 - \lambda)(-1 - \lambda)(8 - \lambda)(-3 - \lambda)$ , which is zero exactly when  $\lambda = 3, -1, 8,$  or  $-3$ .

# A Matrix is Invertible if and only if Zero is not an Eigenvalue

**Fact:**  $A$  is invertible if and only if 0 is not an eigenvalue of  $A$ .

**Why?**

0 is an eigenvalue of  $A \iff Ax = 0x$  has a nontrivial solution

$\iff Ax = 0$  has a nontrivial solution

$\iff A$  is not invertible.

invertible matrix theorem





## Eigenvectors with Distinct Eigenvalues are Linearly Independent

**Fact:** If  $v_1, v_2, \dots, v_k$  are eigenvectors of  $A$  with *distinct* eigenvalues  $\lambda_1, \dots, \lambda_k$ , then  $\{v_1, v_2, \dots, v_k\}$  is linearly independent.

**Why?** If  $k = 2$ , this says  $v_2$  can't lie on the line through  $v_1$ .

But the line through  $v_1$  is contained in the  $\lambda_1$ -eigenspace, and  $v_2$  does not have eigenvalue  $\lambda_1$ .

**In general:** see §5.1 (or work it out for yourself; it's not too hard).

**Consequence:** An  $n \times n$  matrix has at most  $n$  distinct eigenvalues.

# The Invertible Matrix Theorem

## Addenda

We have a couple of new ways of saying “ $A$  is invertible” now:

### The Invertible Matrix Theorem

Let  $A$  be a square  $n \times n$  matrix, and let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be the linear transformation  $T(x) = Ax$ . The following statements are equivalent.

1.  $A$  is invertible.
2.  $T$  is invertible.
3. The reduced row echelon form of  $A$  is  $I_n$ .
4.  $A$  has  $n$  pivots.
5.  $Ax = 0$  has no solutions other than the trivial one.
6.  $\text{Nul}(A) = \{0\}$ .
7.  $\text{nullity}(A) = 0$ .
8. The columns of  $A$  are linearly independent.
9. The columns of  $A$  form a basis for  $\mathbf{R}^n$ .
10.  $T$  is one-to-one.
11.  $Ax = b$  is consistent for all  $b$  in  $\mathbf{R}^n$ .
12.  $Ax = b$  has a unique solution for each  $b$  in  $\mathbf{R}^n$ .
13. The columns of  $A$  span  $\mathbf{R}^n$ .
14.  $\text{Col } A = \mathbf{R}^n$ .
15.  $\dim \text{Col } A = n$ .
16.  $\text{rank } A = n$ .
17.  $T$  is onto.
18. There exists a matrix  $B$  such that  $AB = I_n$ .
19. There exists a matrix  $B$  such that  $BA = I_n$ .
20. The determinant of  $A$  is *not* equal to zero.
21. The number 0 is *not* an eigenvalue of  $A$ .

## Summary

- ▶ **Eigenvectors** and **eigenvalues** are the most important concepts in this course.
- ▶ Eigenvectors are by definition nonzero; eigenvalues may be zero.
- ▶ The eigenvalues of a triangular matrix are the diagonal entries.
- ▶ A matrix is invertible if and only if zero is not an eigenvalue.
- ▶ Eigenvectors with distinct eigenvalues are linearly independent.
- ▶ The  $\lambda$ -eigenspace is the set of all  $\lambda$ -eigenvectors, plus the zero vector.
- ▶ You can compute a basis for the  $\lambda$ -eigenspace by finding the parametric vector form of the solutions of  $(A - \lambda I_n)x = 0$ .