MATH 1553, EXAM 1 SOLUTIONS
SPRING 2023

Name		**GT ID**	

Circle your lecture below.

Jankowski, lecture A (8:25-9:15 AM)   Jankowski, lecture D (9:30-10:20 AM)

Sane, lecture G (12:30-1:20 PM)

Sun, lecture I (2:00-2:50 PM)   Sun, lecture M (3:30-4:20 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, February 8.

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1. TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) Suppose we are given a consistent system of 1 linear equation in 3 variables and the corresponding augmented matrix has 1 pivot in its reduced row echelon form. Then the set of solutions to the equation must be a line.

TRUE | **FALSE**

b) The following vector equation is consistent for every b in $\mathbf{R}^3$:

$$x_1\begin{pmatrix}1\\0\\3\end{pmatrix}+x_2\begin{pmatrix}0\\1\\0\end{pmatrix}+x_3\begin{pmatrix}0\\2\\3\end{pmatrix}=b.$$

TRUE | FALSE

c) Suppose u, v, and w are vectors in $\mathbf{R}^3$. Then $\text{Span}\{u,v,w\}$ is either a plane in $\mathbf{R}^3$ or all of $\mathbf{R}^3$.

TRUE | **FALSE**

d) If A is an $m\times n$ matrix with more columns than rows, then $Ax=b$ must be inconsistent for some b in $\mathbf{R}^m$.

TRUE | **FALSE**

e) Suppose A is a 2×2 matrix and b is a vector in $\mathbf{R}^2$. If the solution set to $Ax=b$ is the span of $\begin{pmatrix}-3\\1\end{pmatrix}$, then $b=\begin{pmatrix}0\\0\end{pmatrix}$.

TRUE | FALSE

Solution.

a) False. This goes back to section 1.1, where we saw that this scenario describes the implicit equation of a plane in $\mathbf{R}^3$. As an illustration, take $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \end{array}\right)$ which corresponds to the implicit equation of the plane $x + 2y + 3z = 1$. Alternatively, we could use section 1.3: one pivot but three variables means that there are two free variables, so the solution set is a plane.

b) True. We could use row-reduction to note that no matter what $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is, the corresponding augmented system will be consistent:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 2 & b_2 \\ 3 & 0 & 3 & b_3 \end{array}\right) \xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & b_1 \\ 0 & \boxed{1} & 2 & b_2 \\ 0 & 0 & \boxed{3} & b_3 - 3b_1 \end{array}\right)$$

Alternatively, in the theme of section 2.3, we could note that the matrix A whose columns are the vectors $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ has a pivot in every row and therefore $Ax = b$ is consistent for all b in $\mathbf{R}^3$.

c) False, the span could be a point or a line. This was done in our 2.1-2.2 worksheet #2.

d) False. If A has a pivot in every row (which is possible) then the system will be consistent for each b in $\mathbf{R}^m$. For example, if A is the 2×3 matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ then $Ax = b$ is consistent for every b in $\mathbf{R}^2$.

e) True. This is #1e from Sample Midterm 1A just slightly rephrased, and it was originally inspired by 2.4 Webwork #3. The span of $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ includes the zero vector, so $x = 0$ is a solution and therefore $b = A(0) = 0$.

2. Multiple choice and short answer. You do not need to show work or justify your answers. Parts (a), (b), and (c) are unrelated.

a) (3 points) Which of the following equations are linear equations in the variables x, y, and z? Clearly circle LINEAR or NOT LINEAR in each case.

(i) $x - yz = 1$ LINEAR **[NOT LINEAR]**

(ii) $9x - 5y + 17z = 7$ **[LINEAR]** NOT LINEAR

(iii) $2x - y + \sqrt{z} = 0$ LINEAR **[NOT LINEAR]**

b) (4 points) Which of the following matrices are in reduced row echelon form (RREF)? Clearly circle all that apply.

[(i) $\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ **]**

[(ii) $\left(\begin{array}{ccc|c} 0 & 1 & 2 & 3 \end{array}\right)$ **]**

(iii) $\left(\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

(iv) $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$

c) (3 points) Write a *vector equation* that corresponds to a system of 2 linear equations in 2 variables (x_1 and x_2) with **infinitely many solutions**.

Solution.

The two vectors on the left *must*:

1. The two vectors on the left side *must* be in $\mathbf{R}^2$, so that they represent a system of two linear equations.
2. The two vectors on the left side *must* be collinear (i.e. one must be a scalar multiple of the other) so that the system will have infinitely many solutions.
3. The vector on the right side must be in the span of the two vectors on the left side, so that the system is consistent. (one easy way to do this is to make this vector the zero vector)

There are many possibilities. For example,

$$x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

or

$$x_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}.$$

One easy example, which is also legitimate, is

$$x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

3. Short answer and multiple choice. Briefly show your work in part (a). Parts (a), (b), (c), and (d) are unrelated.

a) (2 points) Compute the product $\begin{pmatrix} 1 & -1 & -5 \\ 3 & 4 & -7 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

Solution.

This is #4a from Sample Midterm 1A with changed numbers.

$$0\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 4 \end{pmatrix} + 1\begin{pmatrix} -5 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix} + \begin{pmatrix} -5 \\ -7 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ -15 \end{pmatrix}}.$$

b) (2 pts) Write a set of three *different* vectors u, v, and w whose span is a line in $\mathbf{R}^3$.

Solution.

This problem was copied and pasted from 2.1-2.2 Worksheet #2a. Many examples are possible. One way to do this is to choose three different multiples of the same vector:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}.$$

It's also fine to choose one of the three vectors to be the zero vector.

c) (4 points) Suppose A is a matrix and b is a vector in $\mathbf{R}^4$ so that the set of solutions to $Ax = b$ has the parametric vector form given below, where x_3 is a free variable:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

(i) Which of the following are solutions to the equation $Ax = b$? Circle all that apply.

This problem is basically Sample Midterm 1A #3c with changed numbers. It is also similar to Sample Midterm 1B #3c.

$$\boxed{\text{(I)} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} \qquad \text{(II)} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Note $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ which solves $Ax = b$, whereas $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ solves $Ax = 0$.

(ii) How many rows does A have? Circle your answer below.
We were told that b is in $\mathbf{R}^4$, so A must have 4 rows.

1   2   3 $\boxed{4}$ not enough information

(iii) How many columns does A have? Circle your answer below. The solution set lives in $\mathbf{R}^3$, so A must have 3 columns.

1   2 $\boxed{3}$ 4   not enough information

d) (2 points) Let v_1, v_2, w be vectors in $\mathbf{R}^3$, and suppose that the matrix whose three columns are v_1, v_2, and w has three pivots. Which of the following statements must be true? Clearly circle all that apply.

(i) Every vector in $\mathbf{R}^3$ is a linear combination of v_1, v_2, and w.

(ii) w is not a linear combination of v_1 and v_2.

Solution.

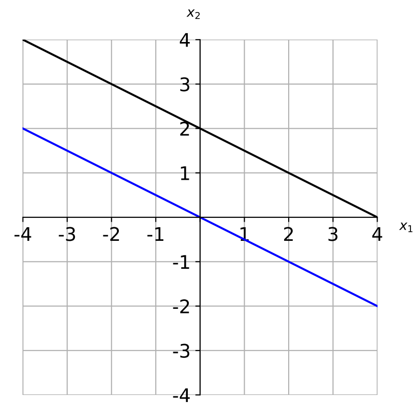
We were told that the matrix $A = \begin{pmatrix} v_1 & v_2 & w \end{pmatrix}$ has a pivot in every row, therefore its columns span $\mathbf{R}^3$ which is just another way of saying statement (i).

Also, since A has three pivots, the system $\left(\begin{array}{cc|c} v_1 & v_2 & w \end{array}\right)$ has a pivot in the final column, therefore the system $x_1v_1 + x_2v_2 = w$ is inconsistent which is just another way of saying statement (ii).

4. **a)** (3 points) For some matrix A and some vector b, the diagonal line below is the solution set for $Ax = b$. On the same graph, draw the solution set for the homogeneous system $Ax = 0$.

Solution.

From section 2.4, the solution set to $Ax = 0$ is parallel to the solution set to $Ax = b$ and that it passes through the origin (since one solution set is a translation of the other). Similar to #2 the 2.3-2.4 worksheet and #7b on Sample Midterm 1A.



b) (4 points) Suppose we are given a system of 3 linear equations in 3 variables. Which of the following statements are true? Clearly circle all that apply.

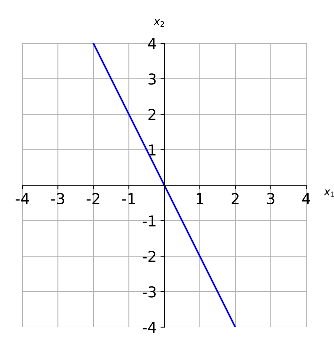
(i) If the system is consistent, then it must have exactly one solution.

(ii) The system must be consistent if its corresponding augmented matrix has 3 pivots.

(iii) The system must be consistent if the RREF of the corresponding augmented matrix has *bottom row* equal to $\begin{pmatrix} 0 & 0 & 1 & | & -2 \end{pmatrix}$.

(iv) One solution to the system must be the trivial solution.

c) (3 points) Let $A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$. Graph the span of the columns of A below. Make sure your graph is clear and complete! This is similar to problems on the practice exams, supplemental problems, and the 2.3-2.4 worksheet. This is the line through the origin containing $(1, -2)$.



The rest of the exam is free response. Unless told otherwise, show your work!

5. The two parts of this problem are unrelated.

a) (5 pts) Consider the linear system of equations given by

$$3x + 10y = 8$$
$$9x - hy = k$$

Find all values of h and k (if there are any) so that the system has **no solution**.

Solution.

This problem is nearly identical to #5a from Sample Midterm 1A and is similar to many problems in worksheets, supplemental problems, and Webwork.

We put the system into an augmented matrix and row reduce.

$$\left(\begin{array}{cc|c} 3 & 10 & 8 \\ 9 & -h & k \end{array}\right) \xrightarrow{R_2=R_2-3R_1} \left(\begin{array}{cc|c} 3 & 10 & 8 \\ 0 & -h-30 & k-24 \end{array}\right)$$

If $-h-30 \neq 0$ then the system will have pivots in every column to the left of the augment bar and thus a unique solution, so we must have $-h-30=0$, thus $\boxed{h=-30}$.

The bottom row of the augmented matrix is $\left(\begin{array}{cc|c} 0 & 0 & k-24 \end{array}\right)$, so for the system to have a pivot in the rightmost column we need $k-24 \neq 0$, thus $\boxed{k \neq 24}$.

b) (5 points) Find all real numbers c so that $\begin{pmatrix} -2 \\ 3 \\ c \end{pmatrix}$ is in Span$\left\{ \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} \right\}$.

Solution.

This problem was basically copied from #5b from Sample Midterm 1A and #4b from Sample Midterm 1B, and the 2.1-2.2 Webwork #6.

We put the system into an augmented matrix and row-reduce.

$$\left(\begin{array}{cc|c} 4 & 5 & -2 \\ 1 & 3 & 3 \\ 6 & 7 & c \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 4 & 5 & -2 \\ 6 & 7 & c \end{array}\right) \xrightarrow[R_3=R_3-6R_1]{R_2=R_2-4R_1} \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -7 & -14 \\ 0 & -11 & c-18 \end{array}\right)$$

$$\xrightarrow{R_2=-R_2/7} \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & -11 & c-18 \end{array}\right) \xrightarrow{R_3=R_3+11R_2} \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & c+4 \end{array}\right).$$

This system is consistent precisely when $c+4=0$, so $\boxed{c=-4}$.

Free response. Show your work in parts (a) and (b). You do not need to show your work on part (c).

6. Consider the following linear system of equations in the variables x_1, x_2, x_3, x_4.

$$x_1 + 4x_2 - x_3 - 4x_4 = 2$$
$$2x_1 + 8x_2 - x_3 - 7x_4 = 4$$
$$5x_1 + 20x_2 - x_3 - 16x_4 = 10.$$

a) (4 points) Write the augmented matrix corresponding to this system, and put the augmented matrix into reduced row echelon form (RREF).

Solution.

Problem 6 may be the most common problem asked on midterm 1 exams in Math 1553. See Sample Midterm 1A #6, Sample Midterm 1B #5, the 2.4 Webwork, the worksheets, and our supplemental practice problems on the course calendar for more examples.

$$\left(\begin{array}{cccc|c} 1 & 4 & -1 & -4 & 2 \\ 2 & 8 & -1 & -7 & 4 \\ 5 & 20 & -1 & -16 & 10 \end{array}\right) \xrightarrow[R_3=R_3-5R_1]{R_2=R_2-2R_1} \left(\begin{array}{cccc|c} 1 & 4 & -1 & -4 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \end{array}\right) \xrightarrow[\text{then } R_1=R_1+R_2]{R_3=R_3-4R_2} \left(\begin{array}{cccc|c} 1 & 4 & 0 & -3 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

b) (4 points) This system is consistent. Write the set of solutions to the system of equations in parametric vector form.

Solution.

We see x_2 and x_4 are free and $x_1 + 4x_2 - 3x_4 = 2$, $x_3 + x_4 = 0$:

$$x_1 = 2 - 4x_2 + 3x_4, \qquad x_2 = x_2 \text{ (real)}, \qquad x_3 = -x_4, \qquad x_4 = x_4 \text{ (real)}.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 - 4x_2 + 3x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -4x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_4 \\ 0 \\ -x_4 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ -1 \\ 1 \end{pmatrix}}.$$

c) (2 points) Write one vector that is *not* the zero vector but is a solution to the corresponding **homogeneous** system of equations given below.

$$x_1 + 4x_2 - x_3 - 4x_4 = 0$$
$$2x_1 + 8x_2 - x_3 - 7x_4 = 0$$
$$5x_1 + 20x_2 - x_3 - 16x_4 = 0.$$

Solution.

The vectors $\begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ are correct answers, as is any nonzero vector in their span. For example, the vector $\begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

Problem 7.

Free response. The two parts of this problem are unrelated. In this problem, we use the usual convention of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to represent vectors in $\mathbf{R}^2$.

a) (6 points) Consider the matrix equation $Ax = b$, where

$$A = \begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

Find the solution set to $Ax = b$ and draw it on the graph below.

Solution.

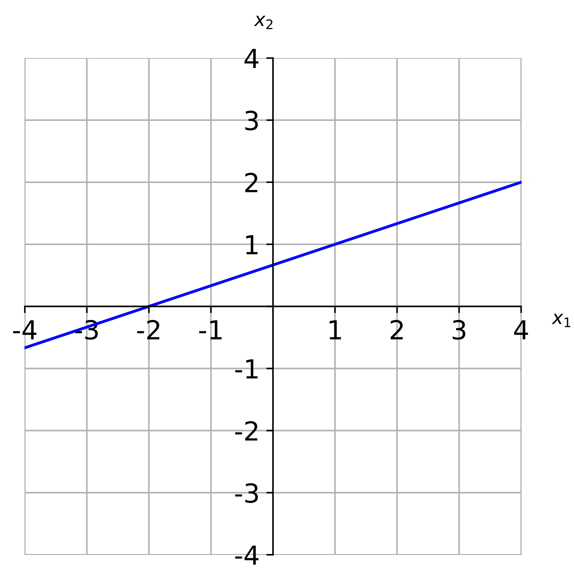
This problem is 2.3-2.4 worksheet's problem #2 slightly modified.

$$\left(A \mid b\right) = \left(\begin{array}{cc|c} -1 & 3 & 2 \\ 2 & -6 & -4 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 = -R_1} \left(\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore, $x_1 = -2 + 3x_2$ and x_2 is free. This is enough to graph the line: when $x_2 = 0$ we get $(-2, 0)$, and when $x_2 = 1$ we get $(1, 1)$, so we just need to graph the line containing those two points. Alternatively we could keep going and use parametric vector form to be clear:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 + 3x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

This is the line through $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ parallel to the span of $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.



b) (4 points) Write an augmented matrix in RREF that corresponds to a system of linear equations in the variables x_1 and x_2 whose solution set has parametric form

$$x_1 = 1 - 4x_2, \qquad x_2 = x_2 \quad (x_2 \text{ free}).$$

Briefly justify why your matrix satisfies these conditions.

Solution.

This is an easier version of #6a from Sample Midterm 1B. We need an augmented matrix with three columns (two to the left of the augment bar, one to the right) that represents $x_1 + 4x_2 = 1$ with x_2 free. Some examples are shown below.

$$\left(\begin{array}{cc|c} 1 & 4 & 1 \end{array}\right), \qquad \left(\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right), \qquad \left(\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.