MATH 1553, SPRING 2023
FINAL EXAMINATION, SOLUTIONS

Name		**GT ID**	

Circle your lecture below.

Jankowski, lec. A and HP (8:25-9:15 AM) Jankowski, lecture D (9:30-10:20 AM)

Sane, lecture G (12:30-1:20 PM)

Sun, lecture I (2:00-2:50 PM) Sun, lecture M (3:30-4:20 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 100 points, and you have 170 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- Simplify your answers as much as possible. For example, you may lose points if you do not simplify $\frac{8}{2}$ to 4, or if you do not simplify $\frac{0.1}{0.9}$ to $\frac{1}{9}$, etc.
- As always, RREF means “reduced row echelon form.” The “zero vector” in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- We use $e_1, e_2, \ldots, e_n$ to denote the standard unit coordinate vectors of $\mathbf{R}^n$.
- We will hand out loose scrap paper, but it **will not be graded**. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 9:00 PM on Tuesday, May 2.

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Problem 1.

a) Suppose W is a subspace of $\mathbf{R}^7$ and that there is a basis of W consisting of 4 vectors. Then $\dim(W) = 4$.

$\boxed{\text{TRUE}}$ FALSE

b) If A is a 5×3 matrix, then the equation $Ax = 0$ must have infinitely many solutions.

TRUE $\boxed{\text{FALSE}}$

c) The set $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - y - z = 3 \right\}$ is a subspace of $\mathbf{R}^3$.

TRUE $\boxed{\text{FALSE}}$

d) Suppose A is an $n \times n$ matrix and $\det(A) = 0$. Then the RREF of A has a row whose entries are all 0.

$\boxed{\text{TRUE}}$ FALSE

e) Let A be a 3×3 diagonalizable matrix. If the only eigenvalue of A is $\lambda = 1$, then A must be the 3×3 identity matrix.

$\boxed{\text{TRUE}}$ FALSE

f) Suppose that $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a matrix transformation $T(x) = Ax$, and suppose that the range of T is a line. Then $\lambda = 0$ must be an eigenvalue of A.

$\boxed{\text{TRUE}}$ FALSE

g) If v and w are eigenvectors of a 3×3 matrix A, then $v + w$ must be an eigenvector of A.

TRUE $\boxed{\text{FALSE}}$

h) Suppose u, v, and w are vectors in $\mathbf{R}^n$. If u is orthogonal to v and u is orthogonal to w, then u must also be orthogonal to $v - w$.

$\boxed{\text{TRUE}}$ FALSE

i) Suppose A is an $m \times n$ matrix and x is a vector in $\mathbf{R}^m$. If x is in the column space of A, then $x \cdot v = 0$ for every vector v in the null space of A^T.

$\boxed{\text{TRUE}}$ FALSE

j) There is a 2×2 positive stochastic matrix A so that $\lambda = i/2$ is an eigenvalue of A.

TRUE $\boxed{\text{FALSE}}$

Solution.

a) True: this comes directly from the definition of dimension.

b) False: the columns of A are three vectors inside $\mathbf{R}^5$, so they may be linearly independent and $Ax = 0$ may have a unique solution. For example,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

c) False: W does not contain the zero vector, since $0 - 0 - 0 \neq 3$.

d) True: since $\det(A) = 0$ we know that A is not invertible so its bottom row in RREF cannot have a pivot. Since we are talking about the RREF, the bottom row can't have a nonzero entry because such an entry would be a pivot.

e) True: if $\lambda = 1$ is the only eigenvalue but A is diagonalizable, then the 1-eigenspace is all of $\mathbf{R}^3$, so $Ax = x$ for all x in $\mathbf{R}^3$, therefore A is the identity matrix.

f) True: we are told that the range of T is a line in $\mathbf{R}^3$. This is just another way of saying that the column space of A is a line in $\mathbf{R}^3$, so A is not invertible, so $\lambda = 0$ is an eigenvalue of A.

g) False: if v and w belong to different eigenspaces then $v + w$ is not an eigenvector of A. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors of A since $Av = v$ and $Aw = 2w$. However, $v + w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not an eigenvector of A since

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

h) True: copied and pasted from the Chapter 6 worksheet. If $u \perp v$ and $u \perp w$ then

$$u \cdot (v - w) = u \cdot v - u \cdot w = 0 - 0 = 0,$$

so $u \perp (v - w)$. In the big picture, this represents the fact that if you want to find all vectors orthogonal to $\mathrm{Span}\{v, w\}$ then you just need to find all vectors that are orthogonal to both v and w.

i) True: $\mathrm{Col}\, A = (\mathrm{Nul}\, A^T)^\perp$, so if x is any vector in the column space of A, then x is orthogonal to every vector in $\mathrm{Nul}\, A^T$.

j) False: A is positive stochastic, so it has $\lambda = 1$ as an eigenvalue. If it had $i/2$ as an eigenvalue then it would also have $-i/2$ as an eigenvalue, so A would have 3 different eigenvalues which is impossible for a 2×2 matrix (an $n \times n$ matrix can never have more than n different eigenvalues).

Problem 2.

Multiple choice and short answer. Parts (a) through (d) are unrelated. You do not need to show your work on this problem, and there is no partial credit.

a) (3 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x - y \geq 0 \right\}$. Determine which properties of a subspace of $\mathbf{R}^2$ are satsified by W to answer the questions below.

(i) Does V contain the zero vector? $\boxed{\text{YES}}$ NO

(ii) Is V closed under addition? In other words, if u and v are vectors in V, must it be true that $u + v$ is also in V? $\boxed{\text{YES}}$ NO

(iii) Is V closed under scalar multiplication? In other words, if c is a real number and u is in V, must it be true that cu is also in V? YES $\boxed{\text{NO}}$

b) Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

(i) (2 points) What is A^{-1}? Clearly circle your answer below.

$$A^{-1} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

$$\boxed{A^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}} \qquad A^{-1} = \frac{1}{7}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \qquad A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

(ii) (1 point) Suppose S is a rectangle in $\mathbf{R}^2$ with area 3, and let T be the matrix transformation $T(x) = Ax$, where A is the matrix in part (i) above.
Fill in the blank: the area of $T(S)$ is $\boxed{21}$.

c) (2 points) Let $B = \begin{pmatrix} 1 & c & 0 \\ 0 & 2 & c \\ 0 & 0 & 1 \end{pmatrix}$, where c is a real number. Which of the following statements must be true? Clearly select all that apply.

$\boxed{\text{(i)}}$ $\text{Col}(B) = \mathbf{R}^3$, no matter what the value of c is.

(ii) If $c = 1$, then B is diagonalizable.

d) (2 points) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$. Which one of the following is a basis for $(\text{Col } A)^{\perp}$?

(I) $\boxed{\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}}$ (II) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ (III) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$ (IV) $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ (V) none of these

Solution.

a) (i) Yes: $0-0=0$.

(ii) Yes:

(iii) No: for example, $u = \begin{pmatrix}1\\0\end{pmatrix}$ is in V since $1-0 \geq 0$, but $-u = \begin{pmatrix}-1\\0\end{pmatrix}$ is not in V since $-1-0<0$.

b) (i) If $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $ad-bc \neq 0$ then $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$, so here

$$A^{-1} = \frac{1}{2(3)-(-1)(1)}\begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix} = \frac{1}{7}\begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix}.$$

You can verify that the above matrix times A will equal I.

(ii) $|\det(A)|\text{Area}(S) = 7(3) = 21$.

c) (i) Yes: no matter what c is, we see B has three pivots so its column span is all of $\mathbf{R}^3$.

(ii) No: if $c=1$ then $A = \begin{pmatrix}1 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 1\end{pmatrix}$, so $(A-I|0) = \left(\begin{array}{ccc|c}0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$. Thus, $A-I$ has two pivots and therefore $\text{Nul}(A-I)=1$. This means $\lambda=1$ only has geometric multiplicity 1 even though it has algebraic multiplicity 2, so A is not diagonalizable.

d) The orthogonal complement of Col A is

$$\text{Nul}\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & -1\end{pmatrix}.$$

Row-reducing $\left(\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 1 & 1 & -1 & 0\end{array}\right)$ gives $\left(\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & -2 & 0\end{array}\right)$ so $x_1 = -x_3$, $x_2 = 2x_3$, and x_3 is free. Therefore, a basis is

$$\left\{\begin{pmatrix}-1\\2\\1\end{pmatrix}\right\}.$$

Alternatively: since A clearly has 2 pivots, its column space is 2-dimensional so $(\text{Col } A)^{\perp}$ has dimension 1 and we can just try (I) and (II) to see if they are orthogonal to both columns of A. Sure enough,

$$\begin{pmatrix}-1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} = -1+1=0, \qquad \begin{pmatrix}-1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix} = -1+2-1=0,$$

so (I) is correct.

Problem 3.

Multiple choice and short answer. Parts (a), (b), and (c) are unrelated. You do not need to show your work on this problem, and there is no partial credit.

a) (2 points) Suppose A is a 25×20 matrix and the dimension of the column space of A is 6. Which *one* of the following describes the null space of A?

(i) $\mathrm{Nul}(A)$ is a 19-dimensional subspace of $\mathbf{R}^{25}$.

(ii) $\mathrm{Nul}(A)$ is a 14-dimensional subspace of $\mathbf{R}^{25}$.

(iii) $\mathrm{Nul}(A)$ is a 19-dimensional subspace of $\mathbf{R}^{20}$.

(iv) $\boxed{\mathrm{Nul}(A) \text{ is a 14-dimensional subspace of } \mathbf{R}^{20}.}$

b) (4 points) Suppose A is a 4×4 matrix with characteristic polynomial

$$\det(A - \lambda I) = (\lambda - 1)^2(\lambda + 2)^2.$$

Which of the following statements are true? Clearly circle all that apply.

$\boxed{\text{(i)}}$ The columns of A are linearly independent.

$\boxed{\text{(ii)}}$ $\det(A) = 4$.

$\boxed{\text{(iii)}}$ If the 1-eigenspace of A is a line, then A is not diagonalizable.

$\boxed{\text{(iv)}}$ The matrix equation $Ax = -2x$ has infinitely many solutions.

c) (4 points) Let A be an 5×4 matrix and let T be the matrix transformation $T(x) = Ax$. Which of the following conditions *guarantee* that T is one-to-one? Clearly circle all that apply.

(i) For each x in $\mathbf{R}^4$, there is at most one vector y in $\mathbf{R}^5$ so that $T(x) = y$.

$\boxed{\text{(ii)}}$ For each y in $\mathbf{R}^5$, there is at most one vector x in $\mathbf{R}^4$ so that $T(x) = y$.

$\boxed{\text{(iii)}}$ The set $\{Ae_1, Ae_2, Ae_3, Ae_4\}$ is linearly independent.

$\boxed{\text{(iv)}}$ The homogeneous matrix equation $Ax = 0$ has only the trivial solution.

Solution.

a) The answer is (iv): Nul(A) lives in $\mathbf{R}^{20}$ since A has 20 columns, and by the Rank Theorem,

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 20, \qquad 6 + \dim(\text{Nul } A) = 20, \qquad \dim(\text{Nul } A) = 14.$$

b) We're given a matrix A with eigenvalues 1 and -2.
(i) Yes: A is invertible since 0 is not an eigenvalue of A, so the columns of A are linearly independent.
(ii) Yes: $\det(A) = \det(A - 0I) = (-1)^2(2)^2 = 4$.
(iii) Yes: this would mean $\lambda = 1$ has algebraic multiplicity 2 but only geometric multiplicity 1, so A is not diagonalizable.
(iv) Yes: this is just the statement that -2 is an eigenvalue of A.

c) (i) No: this is a slight modification of the definition of transformation.
(ii) Yes: this is the definition of one-to-one.
(iii) Yes: this is just the statement that the columns of A are linearly independent, which is an equivalent condition to the transformation being one-to-one.
(iv) Yes: this is an equivalent condition to the transformation being one-to-one.

Problem 4.

Short answer and multiple choice. Parts (a), (b), (c), and (d) are unrelated. You do not need to show your work on this problem.

a) (3 points) Match each matrix below with its corresponding transformation (choosing from (i) through (viii)) by clearly writing that roman numeral next to the matrix. Note there are three matrices and eight options, so not every option is used.

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard matrix for (vii)

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the standard matrix for (iii)

$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is the standard matrix for (vi)

(i) Reflection across the x-axis in $\mathbf{R}^2$.
(ii) Reflection across the y-axis in $\mathbf{R}^2$.
(iii) Reflection across the line $y = x$ in $\mathbf{R}^2$.
(iv) Reflection across the line $y = -x$ in $\mathbf{R}^2$.
(v) Rotation counterclockwise by $\pi/4$ radians in $\mathbf{R}^2$.
(vi) Rotation clockwise by $\pi/4$ radians in $\mathbf{R}^2$.
(vii) Rotation counterclockwise by $\pi/2$ radians in $\mathbf{R}^2$.
(viii) Rotation clockwise by $\pi/2$ radians in $\mathbf{R}^2$.

b) (3 points) Which of the following linear transformations are onto? Circle all that apply.

(i) $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y) = (x + y, x + y)$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $T(x, y) = (x, y, x - y)$.

$\boxed{\text{(iii)}}$ The transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that reflects vectors across the x-axis.

c) (2 points) Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1/4 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}^{-1}$. Find $A\begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

Enter your answer in the space to the right: $A\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \end{pmatrix}$.

d) (2 points) Suppose A is the matrix for the orthogonal projection onto Span$\{v\}$, where v is a nonzero vector in $\mathbf{R}^4$. Which of the following are true? Clearly circle all that apply.

$\boxed{\text{(i)}}$ $Av = v$.

$\boxed{\text{(ii)}}$ The null space of A is 3-dimensional.

Solution.

a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is (vii): counterclockwise rotation $\pi/2$ radians.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is (iii): reflection across the line $y = x$.

$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is (vi): rotation clockwise by $\pi/4$ radians.

b) (i) No: the corresponding matrix A has 2 rows but only 1 pivot. Alternatively, we can just look at the formula for T and see that every vector in the range of T has first entry equal to second entry, so for example $(0,1)$ is not in the range of T. Trying to solve $T(x,y) = (0,1)$ would give the inconsistent system of two equations

$$x + y = 0$$
$$x + y = 1.$$

(ii) No: without doing any work, we see the matrix A for T will have 3 rows and 2 columns, so it cannot have a pivot in every row.

(iii) Yes: its matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which is invertible, so T is onto.

c) We have been given a diagonalization of A where $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is an eigenvector of A for eigenvalue $\lambda = 3$. In other words,

$$A\begin{pmatrix} 1 \\ 5 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \end{pmatrix}.$$

d) Here, A is the matrix for orthogonal projection onto the 1-dimensional subspace of $\mathbf{R}^4$ given by $W = \text{Span}(u)$.

(i) Yes: $Ax = x$ for all x in $W = \text{Span}(v)$.

(ii) Yes: $\dim(\text{Nul } A) = \dim(W^{\perp}) = 3$ since W is a 1-dimensional subspace of $\mathbf{R}^4$.

Problem 5.

Short answer and multiple choice. Parts (a), (b), (c), and (d) are unrelated. You do not need to show your work on this problem, and there is no partial credit.

a) Let $S : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation given by

$$S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-2y \\ x-y-z \end{pmatrix},$$

and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects vectors across the line $y = x$.

(i) (1 point) Let A be the standard matrix for S. What is the dimension of the null space of A? Clearly circle your answer below.

0 $\boxed{1}$ 2 3

(ii) (1 point) Let B be the standard matrix for the composition $T \circ S$. How many columns does B have? Clearly circle your answer below.

1 2 $\boxed{3}$ 4

b) (2 points) Find the value of c so that $\begin{pmatrix} 8 \\ -2 \\ c \end{pmatrix}$ and $\begin{pmatrix} 1 \\ c \\ 3 \end{pmatrix}$ are orthogonal.

Fill in the blank: $c = \boxed{-8}$.

c) (3 points) Suppose A is a 4×4 matrix with 4 **different** real eigenvalues. Which of the following must be true? Clearly circle all that apply.

$\boxed{\text{(i)}}$ Each eigenvalue of A has geometric multiplicity 1.

(ii) Every nonzero vector in $\mathbf{R}^4$ is an eigenvector of A.

$\boxed{\text{(iii)}}$ A is diagonalizable.

d) (3 points) Suppose A and B are $n \times n$ matrices. Which of the following are true? Clearly circle all that apply.

(i) If $Ax = 0$ has only the trivial solution, then $\det(A) = 0$.

(ii) If A and B are invertible, then so is $A+B$ and $(A+B)^{-1} = A^{-1} + B^{-1}$.

$\boxed{\text{(iii)}}$ If $Ax = b$ has infinitely many solutions for some b in $\mathbf{R}^n$, then A is not invertible.

Solution.

a) (i) The matrix A has two pivots:

$$A = \begin{pmatrix} S(e_1) & S(e_2) & S(e_3) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 1 & -1 & -1 \end{pmatrix},$$

so the rank of A is 2 and its null space has dimension 1.
(ii) The matrix for T is 2×2 and the matrix for S is 2×3, so the matrix for $T \circ S$ is 2×2 times a 2×3, which is thus 2×3. Therefore, B has 3 columns.

b) We solve $0 = \begin{pmatrix} 8 \\ -2 \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ c \\ 3 \end{pmatrix} = 8(1) - 2c + 3c = 8 + c$, so $c = -8$.

c) (i) Yes: A is 4×4 with 4 different real eigenvalues, so each eigenvalue has algebraic multiplicity 1 and thus automatically has geometric multiplicity 1.

(ii) No: for example $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ has 4 different real eigenvalues but

$A \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is not an eigenvector.

(iii) Yes: since different eigenvalues yield linearly independent eigenvectors, we get 4 linearly independent eigenvectors in $\mathbf{R}^4$ so A is diagonalizable.

d) (i) No: if $Ax = 0$ has only the trivial solution then A is invertible so $\det(A) \neq 0$.

(ii) No: this is almost never true. For example, $(I + I)^{-1} = \frac{1}{2}I$ but $I^{-1} + I^{-1} = 2I$.

(iii) Yes: this means $Ax = 0$ has infinitely many solutions, so A is not invertible.

Problem 6.

Parts (a), (b), and (c) are unrelated and are 5 points, 3 points, and 2 points, respectively. You do not need to show your work on this problem.

a) Suppose W is a subspace of $\mathbf{R}^3$ and x is a vector in $\mathbf{R}^3$ whose orthogonal decomposition with respect to W is $x = x_W + x_{W^\perp}$ where

$$x_W = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad \text{and} \quad x_{W^\perp} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

(i) (2 points) What is the closest vector to x in W?

(I) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (II) $\frac{1}{\sqrt{17}}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ (III) $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ (IV) $\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$

(V) $\boxed{\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}}$ (VI) $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ (VII) $\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$ (VIII) $\begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix}$

(ii) (2 points) What is the length of x?

(I) $\sqrt{17}$ (II) 9 (III) $\boxed{\sqrt{26}}$ (IV) 7

(V) 5 (VI) 17 (VII) 26 (VIII) 3

(iii) (1 pt) Is $\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ in $W^\perp$? $\boxed{\text{YES}}$ NO NOT ENOUGH INFO

b) (3 points) Suppose A is the matrix for the orthogonal projection onto the subspace $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; 2x_1 - x_2 - x_3 = 0 \right\}$. Which of the following must be true? Clearly select all that apply.

(i) $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$\boxed{\text{(ii)}}$ $A^2 = A$.

$\boxed{\text{(iii)}}$ The eigenvalues of A are 0 and 1.

c) (2 pts) Let A be a 2×2 positive stochastic matrix satisfying $A\begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$.

Fill in the blank: As n gets very large, $A^n\begin{pmatrix} 8 \\ 16 \end{pmatrix}$ approaches the vector $\boxed{\begin{pmatrix} 21 \\ 3 \end{pmatrix}}$.

Solution.

a) (i) The answer is (V): the closest vector to x in W is x_W, which is $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$.

(ii) The answer is (III): $x = x_W + x_{W^\perp} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$, so

$$||x|| = \sqrt{5^2 + 0^2 + 1^2} = \sqrt{26}.$$

(iii) Yes: $W^\perp$ is a subspace, so it is closed under multiplication, and $\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = -x_{W^\perp}$, thus it is in $W^\perp$.

b) (i) No: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is in W since $2(1)-1-1 = 0$, and $Ax = x$ for all x in W, so $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(ii) Yes, this is a property of all orthogonal projections.

(iii) Yes, since A is a projection which is neither the 0 matrix or the identity matrix, its eigenvalues are 0 and 1. Its 1-eigenspace is W and its 0-eigenspace is $W^\perp$.

c) Since A is positive-stochastic and $A\begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$, we conclude that the steady state vector is $w = \frac{1}{8}\begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/8 \\ 1/8 \end{pmatrix}$.

Therefore, as n gets very large, $A^n\begin{pmatrix} 8 \\ 16 \end{pmatrix}$ approaches $(8+16) \cdot \begin{pmatrix} 7/8 \\ 1/8 \end{pmatrix} = \begin{pmatrix} 21 \\ 3 \end{pmatrix}$.

Problem 7.

Free response. You do not need to show your work on parts (a) and (d) of this problem, but show your work on part (b) and justify your answer in part (c).

For this problem, consider the following matrix A and its reduced row echelon form.

$$A = \begin{pmatrix} 1 & -2 & 0 & -1 \\ 2 & -4 & 1 & 1 \\ -2 & 4 & -2 & -4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) (2 points) Write a basis for Col(A).

b) (4 points) Find a basis for Nul(A).

c) (2 points) Write vectors x and y (with $x \neq y$) satisfying $Ax = Ay$. Briefly justify your answer.

d) (2 points) Write a matrix B so that $\text{Col}(B) = \text{Nul}(A)$.

Solution.

a) The pivot columns are guaranteed to form a basis for Col A, so $\left\{\begin{pmatrix}1\\2\\-2\end{pmatrix},\begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}$ is a basis. However, for this particular matrix A, any choice of two columns of A will form a basis of A unless we choose the first two columns (which are scalar multiples of each other).

b) From the RREF of $(A|0)$ we see $x_1-2x_2-x_4=0$ and $x_3+3x_4=0$, where x_2 and x_4 are free. Therefore, $x_1=2x_2+x_4$ and $x_3=-3x_4$, so

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}2x_2+x_4\\x_2\\-3x_4\\x_4\end{pmatrix}=\begin{pmatrix}2x_2\\x_2\\0\\0\end{pmatrix}+\begin{pmatrix}x_4\\0\\-3x_4\\x_4\end{pmatrix}=x_2\begin{pmatrix}2\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}1\\0\\-3\\1\end{pmatrix}.$$

One basis for Nul A is $\left\{\begin{pmatrix}2\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\-3\\1\end{pmatrix}\right\}$.

c) Many possibilities. We can just take two vectors in the null space of A, for example, say $x=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ and $y=\begin{pmatrix}2\\1\\0\\0\end{pmatrix}$.

d) We just need a matrix whose columns span Nul A. This is as simple as $B=\begin{pmatrix}2&1\\1&0\\0&-3\\0&1\end{pmatrix}$.

Problem 8.

Free response. Show your work unless instructed otherwise! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) (5 points) Let $A = \begin{pmatrix} -2 & -4 \\ 8 & 6 \end{pmatrix}$. Find the eigenvalues of A. For the eigenvalue with *negative* imaginary part, find one corresponding eigenvector v. Enter your answers in the space provided below.

The eigenvalues are ______________________. [simplify the eigenvalues completely]

For the eigenvalue with *negative* imaginary part, an eigenvector is $v = \begin{pmatrix} \quad \end{pmatrix}$.

b) (5 points) Let $W = \text{Span}\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\}$, and let $x = \begin{pmatrix} 15 \\ 15 \end{pmatrix}$. Find the orthogonal decomposition of x with respect to W. In other words, find x_W in W and $x_{W^\perp}$ in $W^\perp$ so that $x = x_W + x_{W^\perp}$. Write your answers in the space provided below.

$$x_W = \begin{pmatrix} \quad \\ \quad \end{pmatrix} \qquad x_{W^\perp} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}.$$

Solution.

a) The characteristic equation is

$$\det\begin{pmatrix} -2-\lambda & -4 \\ 8 & 6-\lambda \end{pmatrix} = 0, \qquad (-2-\lambda)(6-\lambda)+32=0, \qquad \lambda^2-4\lambda+20=0.$$

Alternatively, we could have used the shortcut

$$0=\lambda^2-\mathrm{Tr}(A)\lambda+\det(A)=\lambda^2-4\lambda+20.$$

The eigenvalues are

$$\lambda=\frac{4\pm\sqrt{4^2-4(20)}}{2}=\frac{4\pm\sqrt{-64}}{2}=\frac{4\pm 8i}{2}=2\pm 4i.$$

For the eigenvalue $2-4i$, we use the 2×2 eigenvector trick for the first row $(a\ \ b)$ of $A-\lambda I$:

$$\left(A-(2-4i)I \mid 0\right)=\left(\begin{array}{cc|c} -4+4i & -4 & 0 \\ (*) & (*) & 0 \end{array}\right),$$

so an eigenvector is $\begin{pmatrix} -b \\ a \end{pmatrix}=\begin{pmatrix} 4 \\ -4+4i \end{pmatrix}$.

Another possibility is $\begin{pmatrix} b \\ -a \end{pmatrix}=\begin{pmatrix} -4 \\ 4-4i \end{pmatrix}$ which is just -1 times the first eigenvector.

Another possibility is to use the trick with the second row:

$\begin{pmatrix} 8 & 6-2(2-4i) \end{pmatrix}=\begin{pmatrix} 8 & 4+4i \end{pmatrix}$ so $v=\begin{pmatrix} -b \\ a \end{pmatrix}=\begin{pmatrix} -4-4i \\ 8 \end{pmatrix}$ or $v=\begin{pmatrix} 4+4i \\ -8 \end{pmatrix}$.

b) With $u=\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, the matrix for orthogonal projection onto W is

$$B=\frac{1}{u\cdot u}uu^T=\frac{1}{(-1)^2+2^2}\begin{pmatrix} -1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \end{pmatrix}=\frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$$

Therefore,

$$x_W=Bx=\begin{pmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{pmatrix}\begin{pmatrix} 15 \\ 15 \end{pmatrix}=\begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

$$x_{W^\perp}=x-x_W=\begin{pmatrix} 15 \\ 15 \end{pmatrix}-\begin{pmatrix} -3 \\ 6 \end{pmatrix}=\begin{pmatrix} 18 \\ 9 \end{pmatrix}.$$

Problem 9.

Show your work unless instructed otherwise! A correct answer without sufficient work may receive little or no credit.

For this problem, let $A = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -3 & 5 \end{pmatrix}$.

a) (2 points) Find all eigenvalues of A and write them in the box below. You do not need to show your work on this part.

b) (5 points) For each of the eigenvalues, find a basis of the corresponding eigenspace.

c) (3 points) A is diagonalizable. Write an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}.$$

Solution.

a) The matrix is lower triangular so the eigenvalues are the diagonal entries, namely $\lambda = 5$ and $\lambda = 2$.

b) 5-eigenspace:

$$(A - 5I|0) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & -3 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $x_1 - 3x_2 = 0$ (so $x_1 = 3x_2$) while x_2 and x_3 are free.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \qquad \text{Basis for 5-eigenspace} : \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

2-eigenspace:

$$(A - 2I|0) = \left(\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 \end{array}\right) \xrightarrow[R_2 = R_2 - R_1,\ R_3 = R_3 - R_1]{R_1 = R_1/3,\text{ then:}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right). \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $x_1 = 0$, $x_2 - x_3 = 0$ (so $x_2 = x_3$), and x_3 is free.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \qquad \text{Basis for 2-eigenspace} : \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

c) We need the columns of C to be a basis of $\mathbf{R}^3$ consisting of eigenvectors, and D to be the diagonal matrix of eigenvalues written in corresponding order. For example,

$$C = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Other examples are possible, for example

$$C = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Problem 10.

Free response. Show your work!

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0, -12), \qquad (2, 7), \qquad (6, 3).$$

Enter your answer below:

$$y = \underline{\quad 2 \quad} x + \underline{\quad -6 \quad}.$$

You must show appropriate work and *simplify your answer completely* (if your answer has fractions, simplify them completely). If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

Solution.

We imagine there was a line through three points.

$x = 0$, $y = -12$: $\quad -12 = M(0) + B \quad 0M + B = -12$

$x = 2$, $y = 7$: $\quad 7 = M(2) + B \quad 2M + B = 7$

$x = 6$, $y = 3$: $\quad 3 = M(6) + B \quad 6M + B = 3$

This is $Ax = b$ where $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 6 & 1 \end{pmatrix}$, $x = \begin{pmatrix} M \\ B \end{pmatrix}$, and $b = \begin{pmatrix} -12 \\ 7 \\ 3 \end{pmatrix}$.

We will need to solve $A^T A \widehat{x} = A^T b$.

$$A^T A = \begin{pmatrix} 0 & 2 & 6 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 6 & 1 \end{pmatrix} = \begin{pmatrix} 40 & 8 \\ 8 & 3 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 0 & 2 & 6 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -12 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 32 \\ -2 \end{pmatrix}.$$

Now we solve for $\widehat{x}$:

$$\left(A^T A \mid A^T b \right) = \left(\begin{array}{cc|c} 40 & 8 & 32 \\ 8 & 3 & -2 \end{array} \right) \xrightarrow[\text{then } R_2 = R_2 - 5R_1]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 8 & 3 & -2 \\ 0 & -7 & 42 \end{array} \right) \xrightarrow{R_2 = -R_2/7} \left(\begin{array}{cc|c} 8 & 3 & -2 \\ 0 & 1 & -6 \end{array} \right)$$

$$\xrightarrow{R_1 = R_1 - 3R_2} \left(\begin{array}{cc|c} 8 & 0 & 16 \\ 0 & 1 & -6 \end{array} \right) \xrightarrow{R_1 = R_1/8} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -6 \end{array} \right).$$

This gives $\widehat{x} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$, and the order is $\begin{pmatrix} M \\ B \end{pmatrix}$ from our setup, so

$$y = 2x - 6.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

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