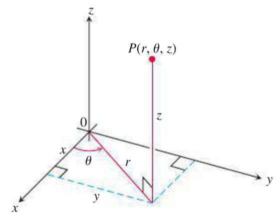
Best Work #4: Selected Materials from Math 2550, Introduction to Multivariable Calculus

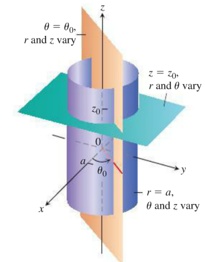
15.7, Integration in Cylindrical and Spherical Coordinates

Cylindrical coordinates

Imagine life on a right circular cylinder.

Your (x, y)-position is described very nicely by polar coordinates, so your (x, y, z)-position can be described by r, θ, z.





r and θ: polar coordinates for the projection onto xy-plane.
z: the rectangular vertical coordinate.

Integration in cylindrical coordinates

Polar: $dx\ dy$ became $r\ dr\ d\theta$.

Cylindrical: as we'd expect, $dx\ dy\ dz$ becomes $r\ dr\ d\theta\ dz$.

So, if $f(x, y, z)$ is a function of three variables,

$$\iiint_D f\ dV = \iiint_C f(r, \theta, z) r\ dz\ \ dr\ d\theta,$$

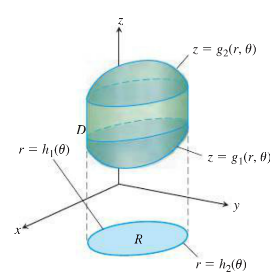
where C is the region D written in cylindrical coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}.$$

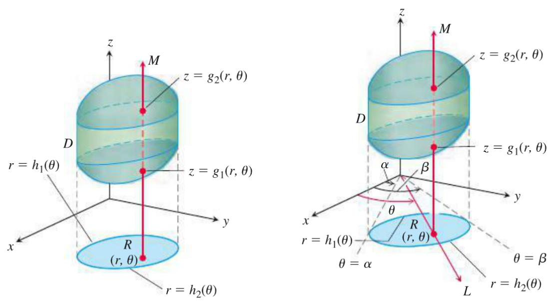
Limits of integration

Often, we take care of the z-limits first: go in direction of increasing z (from entrance to exit).



Limits of integration, continued

Next, we take care of the projection in terms of polar coordinates.

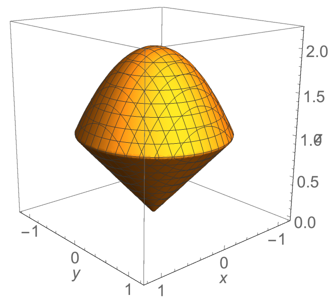


$$\iiint_D f\ dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z) r\ dz\ dr\ d\theta.$$

Example 1

Let D be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integrals in cylindrical coordinates for $\iiint_D (x^2 - z)\ dV$ using the orders:

$(a)\ dz\ dr\ d\theta \qquad (b)\ dr\ dz\ d\theta$

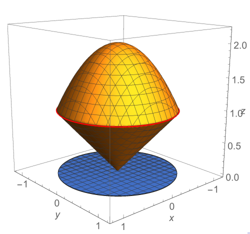


Example 1(a)

(a) Cylindrical coords: $r^2 = x^2 + y^2$, $z = z$. A ray in the z-dir:
Enters at $z = \sqrt{x^2+y^2}$, so $z = \sqrt{r^2} = r$
Exits at $z = 2 - x^2 - y^2$, so $z = 2 - r^2$. The shadow region in the xy-plane lies below the widest part of the region, which is the intersection of the two outer surfaces $z = r$ and $z = 2 - r^2$.

$r = 2 - r^2 \qquad r^2 + r - 2 = 0 \qquad (r-1)(r+2) = 0 \qquad r = 1, \quad \cancel{r = -2}.$

Shadow region: enclosed by $r = 1$, so $0 \le r \le 1$, $0 \le \theta \le 2\pi$.



Example 1(a)

$x^2 - z = r^2\cos^2(\theta) - z$, so our final answer is

$$\iiint_D f\ dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} \left(r^2\cos^2(\theta) - z\right) r\ dz\ dr\ d\theta$$

$$= \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} (r^3\cos^2(\theta) - zr)\ dz\ dr\ d\theta$$

$$= \ldots(\text{with work})\ldots = -\frac{47\pi}{60}.$$

Example 1(b)

For any point P in D, r is the distance from P to the z-axis. Start anywhere on the z-axis and move outward in direction of increasing r (no matter what θ is). We will enter D when $r = 0$.

But where we exit D depends on *where we started on the z-axis*.

- If $0 \leq z \leq 1$, we will exit through the surface $z = \sqrt{x^2 + y^2}$, which is $z = \sqrt{r^2} = r$.
- If $1 \leq z \leq 2$, we will exit through the surface $z = 2 - x^2 - y^2 = 2 - r^2$.

$$z = 2 - r^2 \qquad r^2 = 2 - z \qquad r = \sqrt{2 - z} \quad (r \geq 0).$$

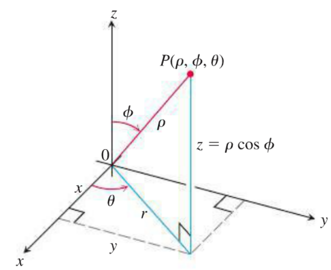
Therefore,

$$\iiint_D f \, dV = \int_0^{2\pi} \int_0^1 \int_0^z \left(r^2 \cos^2(\theta) - z\right) r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} \left(r^2 \cos^2(\theta) - z\right) r \, dr \, dz \, d\theta.$$

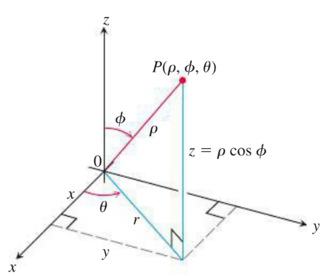
Spherical Coordinates

Spherical coordinates represent a point P by (ρ, ϕ, θ), where:

- ρ is the distance from P to origin ($\rho = \sqrt{x^2 + y^2 + z^2}$, $\rho \geq 0$)
- ϕ is the angle $\overrightarrow{0P}$ makes with the positive z-axis. ($0 \leq \phi \leq \pi$)
- θ is the angle from cylindrical coordinates.



Spherical Coordinates (same picture as previous page)



We see $x = r\cos(\theta)$, $y = r\sin(\theta)$, where $r = \rho\sin(\phi)$, so:

$$x = \rho\sin(\phi)\cos(\theta) \qquad y = \rho\sin(\phi)\sin(\theta) \qquad z = \rho\cos(\phi).$$

Example 2

1. Draw the object given by $\phi = \frac{\pi}{6}$. Write its equation in standard (rectangular) coordinates.
2. Write the equation $x^2 + y^2 + (z-2)^2 = 4$ in spherical coordinates.

Example 2(a)

Solution: (a) By def. of spherical coordinates, this is the set of all P so that $\overrightarrow{0P}$ makes an angle of $\phi = \dfrac{\pi}{6}$ with the positive z-axis.

In other words, start at the origin and travel along a straight line at an angle of $\dfrac{\pi}{6}$ with the positive z-axis. (since $\phi = \frac{\pi}{6}$, the point will have positive z-coordinate).

Imagine doing this for a line in the xz-plane, then do it for the yz-plane, and rotate around the z-axis to get the full picture.

This gives us the top half of a cone. What is its equation in standard (rectangular) coordinates?

Example 2(a), continued

$$x = \rho \sin\left(\frac{\pi}{6}\right)\cos(\theta), \qquad y = \rho \sin\left(\frac{\pi}{6}\right)\sin(\theta) \qquad z = \rho \cos\left(\frac{\pi}{6}\right)$$

$$x = \frac{1}{2}\rho\cos(\theta) \qquad y = \frac{1}{2}\rho\sin(\theta) \qquad z = \frac{\sqrt{3}}{2}\rho$$

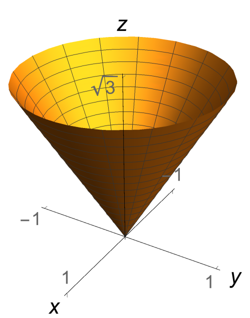
$$x^2 + y^2 = \frac{1}{4}\rho^2\left(\cos^2\theta + \sin^2\theta\right) = \frac{1}{4}\rho^2 \qquad z^2 = \frac{3}{4}\rho^2$$

We see $x^2 + y^2$ is related to z^2.

In fact, $3(x^2 + y^2) = z^2$, so our cone has equation

$$z = \sqrt{3x^2 + 3y^2}.$$

Example 2(a), graph



Note the points $(\pm 1, 0, \sqrt{3})$ and $(0, \pm 1, \sqrt{3})$ on the graph.

Example 2(b)

We are given the sphere $x^2 + y^2 + (z-2)^2 = 4$.

$$x^2 + y^2 + (z-2)^2 = 4.$$

$$x = \rho\sin(\phi)\cos(\theta) \qquad y = \rho\sin(\phi)\sin(\theta) \qquad z = \rho\cos(\phi), \text{ so}$$

$$(\rho\sin(\phi)\cos(\theta))^2 + (\rho\sin(\phi)\sin(\theta))^2 + [\rho\cos(\phi) - 2]^2 = 4,$$

$$\rho^2\sin^2(\phi)\cos^2(\theta) \ + \ \rho^2\sin^2(\phi)\sin^2(\theta) + \left[\rho^2\cos^2(\phi) - 4\rho\cos(\phi) + 4\right] = 4$$

We factor $\rho^2\sin^2(\phi)$ from the first two terms:

$$\rho^2\sin^2(\phi)\left(\cos^2\theta + \sin^2\theta\right) + \rho^2\cos^2(\phi) - 4\rho\cos(\phi) = 0$$

$$\rho^2\sin^2(\phi)(1) + \rho^2\cos^2(\phi) - 4\rho\cos(\phi) = 0 \qquad \rho^2\left[\sin^2(\phi) + \cos^2(\phi)\right] = 4\rho\cos(\phi)$$

$$\rho^2 = 4\rho\cos(\phi) \qquad \rho = 4\cos(\phi) \quad (\rho > 0).$$

When $\phi = \pi/2$ we get $\rho = 0$, putting us at the origin.

Integration in Spherical Coordinates

If D is a region written in spherical coordinates, then the triple integral of a function f over D is

$$\iiint_D f(\rho, \phi, \theta)\rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta.$$

Coordinate Conversion Formulas

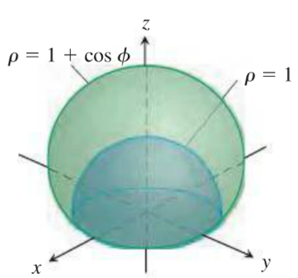
Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

Example 3

Find the volume of the region within $z \geq 0$, between the surface $\rho = 1 + \cos(\phi)$ and the upper hemisphere of radius 1 centered at the origin.



Example 3, solution

We're told $z \geq 0$, so ϕ can only take values between 0 and $\frac{\pi}{2}$.

If we start at the origin and go in a direction making an angle ϕ with the positive z-axis, we will enter the surface when $\rho = 1$ and exit at $\rho = 1 + \cos(\phi)$.

For any given angle θ, we see ϕ sweeps from $\phi_{\text{min}} = 0$ to $\phi_{\text{max}} = \frac{\pi}{2}$.

Also, θ sweeps from 0 to 2π.

Thus, the volume is

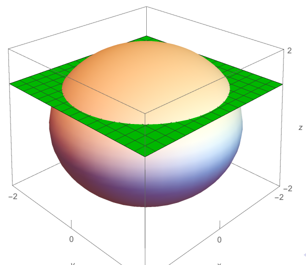
$$\int_0^{2\pi}\int_0^{\pi/2}\int_1^{1+\cos(\phi)} \rho^2\sin(\phi) d\rho\ d\phi\ d\theta.$$

Example 4

The plane $z = 1$ slices the solid sphere $x^2 + y^2 + z^2 \leq 4$ into an upper piece and a lower piece. Write triple integrals that would give the volume of the upper piece D of the sphere using:

1. Spherical coordinates
2. Cylindrical coordinates
3. Rectangular coordinates

Carry out one of these triple integrals.



Example 4(a), solution

(a) $x^2 + y^2 + z^2 \leq 2^2$, so sphere has radius 2, which means $\rho \leq 2$.

ρ limits: Starting at the origin and traveling in the direction that makes an angle ϕ with the positive z-axis, we enter at $z = 1$ and exit at $\rho = 2$. In cylindrical coordinates, $z = 1$ is $\rho\cos(\phi) = 1$, so $\rho = \sec(\phi)$. $\sec(\phi) \leq \rho \leq 2$.

ϕ limits: We see ϕ sweeps from $\phi = 0$ until it hits the intersection of $z = 1$ ($\rho = \sec\phi$) and $x^2 + y^2 + z^2 = 4$ ($\rho = 2$). This gives us $\sec(\phi) = 2$, so $\phi = \frac{\pi}{3}$. Alternatively, we could have used points and triangle geometry to find $0 \leq \phi \leq \frac{\pi}{3}$.

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\phi)}^{2} \rho^2 \sin(\phi) d\rho \, d\phi \, d\theta.$$

Example 4(a), solution

Here is the triple integral, carried out.

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \frac{1}{3}\int_0^{2\pi} \int_0^{\pi/3} \left(8\sin\phi - \tan\phi\sec^2\phi\right) d\phi \, d\theta \\
&= \frac{1}{3}\int_0^{2\pi} \left[-8\cos\phi - \frac{1}{2}\tan^2\phi\right]_0^{\pi/3} d\theta \\
&= \frac{1}{3}\int_0^{2\pi} \left[-4 - \frac{1}{2}(3) + 8\right] d\theta = \frac{1}{3}\int_0^{2\pi} \frac{5}{2} \, d\theta = \frac{5}{6}(2\pi) = \frac{5\pi}{3}
\end{aligned}$$

Example 4(b), solution

(b) For cylindrical coordinates, we need r, θ, and z.

z limits: For any z-ray, the entrance (bottom) is $z = 1$ and the exit (top) is at $x^2 + y^2 + z^2 = 4$:

$x^2 + y^2 + z^2 = 4 \qquad r^2 + z^2 = 4 \qquad z = \sqrt{4 - r^2}$ (since $z \geq 1$).

So $1 \leq z \leq \sqrt{4 - r^2}$.

Shadow region R: The outer contour of intersection of plane and sphere is $z = 1$, so

$$x^2 + y^2 + z^2 = 4 \qquad x^2 + y^2 + 1 = 4 \qquad x^2 + y^2 = 3.$$

Thus R is the region $x^2 + y^2 \leq 3$, the solid circle of radius $\sqrt{3}$ centered at the origin.

Therefore,

$$V = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta.$$

Example 4(b), solution

$$V = \int_0^{2\pi}\int_0^{\sqrt{3}} (r\sqrt{4-r^2} - r)\ dr\ d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^2}{2} \right]_{r=0}^{r=\sqrt{3}} d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}(1) - \frac{3}{2} \right] - \left[-\frac{1}{3}\cdot 4^{3/2} \right] d\theta = \int_0^{2\pi} \left(-\frac{1}{3} - \frac{3}{2} + \frac{8}{3} \right) d\theta$$

$$= \int_0^{2\pi} \frac{5}{6} d\theta = \left[\frac{5}{6}\theta \right]_{\theta=0}^{\theta=2\pi} = \frac{5\pi}{3}.$$

Example 4(c), solution

(c) For rectangular coordinates, we see a z-ray in the positive z-direction enters when $z = 1$ and exits when $x^2 + y^2 + z^2 = 4$, i.e. $z = \sqrt{4 - x^2 - y^2}$. The shadow region is the region below the intersection of $z = 1$ and $x^2 + y^2 + z^2 = 4$, so $x^2 + y^2 = 3$.

Therefore, $-2 \le x \le 2$, $-\sqrt{3 - y^2} \le y \le \sqrt{3 - y^2}$.

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dz\ dy\ dx$$

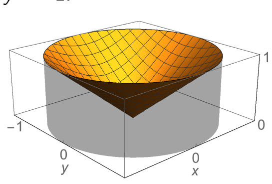
$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \left[\sqrt{4 - x^2 - y^2} - 1 \right] dy\ dx = \text{(now what)?}$$

At this point we would convert to polar coordinates, which would put us right back into the situation of example 4(b).

Example 5

A solid D is bounded below by the plane $z = 0$, bounded above by the cone $z = \sqrt{x^2 + y^2}$, and bounded on the sides by the cylinder $x^2 + y^2 = 1$. The density function is $\delta(x, y, z) = z$. Find the center of mass of the region.

Solution: First, let's graph the region. We want the solid portion between the cone and the cylinder (above $z = 0$). The cone and cylinder intersect when $z = 1$, and the shadow region is enclosed by the circle $x^2 + y^2 = 1$.



Example 5, continued

From geometric symmetry and the fact that the density only depends on z, we see $\overline{x} = \overline{y} = 0$.
(to be sure, we could compute and find $M_{yz} = M_{xz} = 0$)

Convert to cylindrical coords. A z-ray enters at $z = 0$ and exits at $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$. For $\overline{z}$, we compute M and M_{xy}.

$$M = \iiint_D \delta(x,y,z)\ dV = \iiint_D z\ dV = \int_0^{2\pi}\int_0^1\int_0^r z(r\ dz\ dr\ d\theta) = \int_0^{2\pi}\int_0^1 \left[r\frac{z^2}{2} \right]_{z=0}^{z=r} dr\ d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{r^3}{2}\ dr\ d\theta = \int_0^{2\pi} \left[\frac{r^4}{8} \right]_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{1}{8}\ d\theta = \frac{\pi}{4}.$$

Example 5, continued

$$M_{xy} = \iiint_D z\delta(x,y,z)\ dV = \iiint_D z \cdot z\ dV = \int_0^{2\pi}\int_0^1\int_0^r z^2(r\ dz\ dr\ d\theta)$$

$$= \int_0^{2\pi}\int_0^1\int_0^r rz^2\ dz\ dr\ d\theta = \int_0^{2\pi}\int_0^1 \left[r \cdot \frac{z^3}{3}\right]_{z=0}^{z=r} dr\ d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{r^4}{3} dr\ d\theta = \int_0^{2\pi}\left[\frac{r^5}{5\cdot 3}\right]_{r=0}^{r=1} d\theta = \int_0^{2\pi}\frac{1}{15}\ d\theta = \frac{2\pi}{15}.$$

We find

$$\overline{z} = \frac{M_{xy}}{M} = \frac{2\pi/15}{\pi/4} = \frac{2\pi}{15}\cdot\frac{4}{\pi} = \frac{8}{15}.$$

The center of mass is $\left(0, 0, \frac{8}{15}\right)$.

MATH 2550, EXAM 1
SPRING 2019, JANKOWSKI

Name		**Section**	

1	2	3	4	5	Total

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- You exam has 50 points, and you have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will provide loose scrap paper, but the scrap paper will not be collected or graded under any circumstances. Anything that you want to be graded must be written on the exam itself.
- Good luck!

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1. Parts (a)-(c) are 2 points each. You do not need to show your work or justify your answers on parts (a) through (c). Part (d) is worth 4 points.

a) The planes $x-5y+z=2$ and $2x+y+z=1$ are perpendicular.

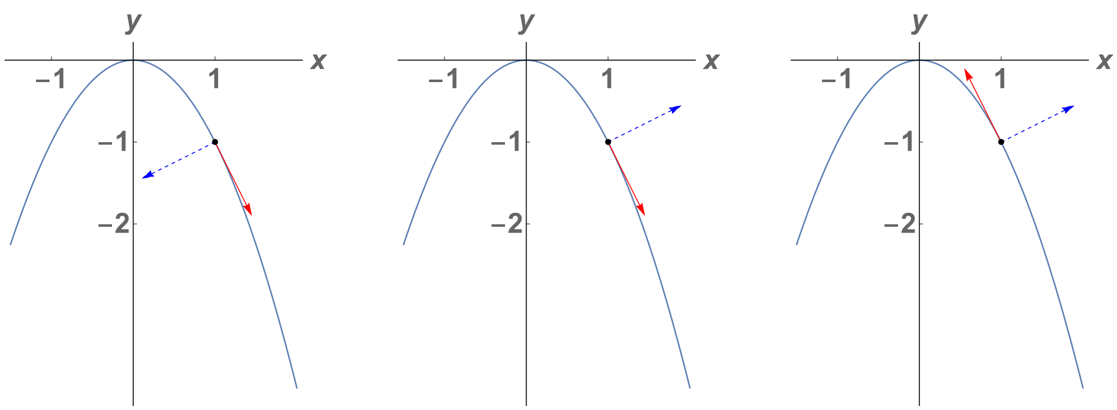
TRUE FALSE

b) If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbf{R}^3$, then $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ is a vector which is orthogonal to $\mathbf{w}$. (recall that two vectors are orthogonal when their dot product is zero)

TRUE FALSE

c) A curve $\mathbf{r}$ is traversed from left to right as t increases. Circle the graph below where, at the point $(1,-1)$:

- The unit tangent $\mathbf{T}$ is correctly graphed as the solid arrow, and
- The principal unit normal $\mathbf{N}$ is correctly graphed as the dashed arrow.



d) Find a vector parallel to the line of intersection of the planes $x-2y+2z=1$ and $x-4y+z=3$.

Space for extra work on problem 1.

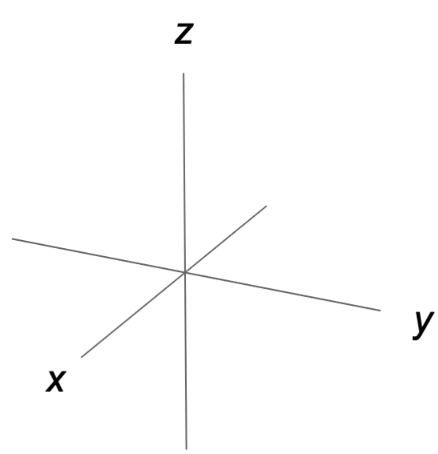
2. **a)** [4 points] Find the distance from $(1,-1,3)$ to the plane $2x - y + 2z = 2$.

b) [2 points] Consider the surface given by the equation $x^2 - y^2 = 1 + z^2$.

The type of quadric surface is: ____________________

c) [4 points] Consider the surface given by the equation $z = 1 - x^2 - y^2$. Identify and sketch the surface below.

The type of quadric surface is: ____________________



Space for extra work on problem 2.

3. **a)** [4 points] Miranda Keyes launches a projectile from 5 feet above ground level at a speed of $64\sqrt{2}$ feet per second, at an angle of 45° measured from the horizontal. It follows ideal projectile motion. Acceleration due to gravity is downward at 32 ft/s^2.

After how many seconds will the projectile reach its maximum height?

b) [5 points] Consider the curve $\mathbf{r}(t) = \langle \sqrt{3}\cos t,\ t,\ \sqrt{3}\sin t\rangle$ for $0 \le t < 2\pi$. Find the point at a distance of π units along the curve starting from $(-\sqrt{3}, \pi, 0)$, traversed in the direction of increasing arclength.

Space for extra work on problem 3.

4. **a)** Consider the smooth ellipse $\mathbf{r}(t) = \langle 3\cos t,\ \sin t\rangle$ for $-\infty < t < \infty$.

(i) [2 pts] Find the formula for the unit tangent vector $\mathbf{T}(t)$.

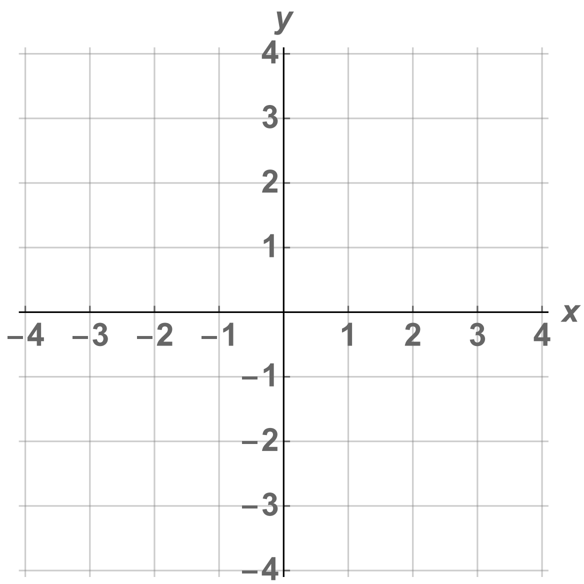
(ii) [4 pts] At the point $(-3,\ 0)$, find $\mathbf{T}$ and the principal unit normal $\mathbf{N}$.

(iii) [4 pts] At $(-3,\ 0)$, the curvature of $\mathbf{r}$ is $\kappa = 3$ (you don't need to calculate this). Find the circle of curvature for $\mathbf{r}$ at $(-3,\ 0)$.

Write your answer in the form $(x-a)^2 + (y-b)^2 = c^2$ (for some a, b, and c)

Space for extra work on problem 4.

5. **a)** [6 points] Let $f(x,y) = \dfrac{\ln(x-y)}{\sqrt{4-x^2-y^2}}$. Graph the domain of f on the axes below. Show your work to indicate how you found the domain of f. Carefully indicate whether the points along a given boundary are in the domain or not (you do NOT need to state whether the domain is open or closed or neither).



b) Let $\mathbf{u} = \langle 3, -1, 1 \rangle$.

(i) [3 pts] If $\mathbf{v}$ is a unit vector, what is the maximum possible value of $\mathbf{u} \cdot \mathbf{v}$? Briefly justify your answer.

(ii) [2 pt] Suppose $\mathbf{v}$ is a unit vector that gives the maximum possible value of $\mathbf{u} \cdot \mathbf{v}$ from part (i). Find $\mathbf{u} \times \mathbf{v}$. Justify your answer either using a geometric argument or a computation.

Space for extra work on problem 5.

MATH 2550, EXAM 2
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- Write your name on the front of each page (not just the cover page!).
- You exam has 50 points, and you have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will provide loose scrap paper, but the scrap paper will not be collected or graded under any circumstances. Anything that you want to be graded must be written on the exam itself.
- Good luck!

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1. Parts (a) and (b) are true or false. If the statement is always true, answer true. Otherwise, answer false. You do not need to show any work on parts (a) and (b).

a) (2 pts) If a function $f(x, y)$ is differentiable at $(0, 0)$, then the directional derivative of f at $(0, 0)$ in the direction of the unit vector $\left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$ is given by

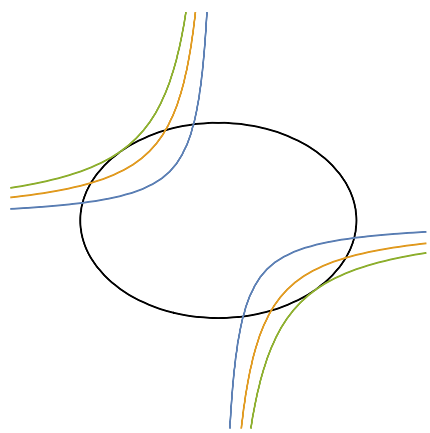
$$\lim_{h\to 0} \frac{f(-2h,\ h) - f(0,0)}{h}. \qquad \text{TRUE} \qquad \text{FALSE}$$

b) (2 pts) The tangent plane for the graph of $f(x, y) = \ln(x^2 - y)$ at the point $(2, 1, \ln(3))$ is given by the equation

$$z = \ln(3) + \frac{2x}{x^2 - y}(x-2) - \frac{1}{x^2 - y}(y-1).$$

TRUE FALSE

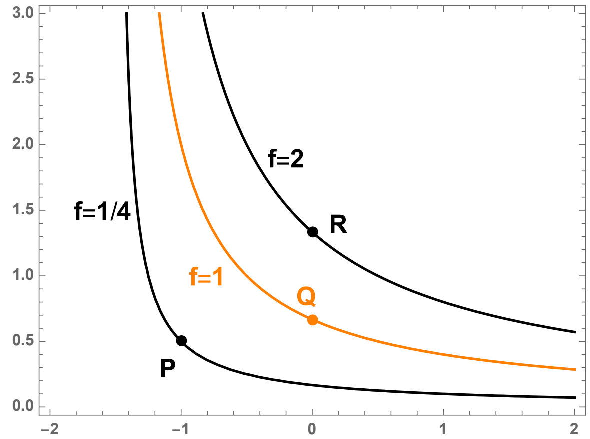
c) (3 pts) Some level curves for a function $f(x, y)$ are given below. The ellipse is the level curve $g(x, y) = 1$ for some smooth function g. Clearly label all points on the ellipse where ∇f is a scalar multiple of ∇g.
(we're assuming $\nabla f \neq \mathbf{0}$ and $\nabla g \neq \mathbf{0}$ and the functions are smooth at all necessary points)



d) (4 points) Suppose $w = f(x, y)$ satisfies $\frac{\partial f}{\partial x} = 3x^2 - 3y$ and $\frac{\partial f}{\partial y} = -3x$, where $x = \sin(2t)$ and $y = \sin(t)$. Find $\frac{dw}{dt}$ when $t = \frac{\pi}{2}$.

Space for extra work on problem 1.

2. **a)** For some smooth $f(x,y)$, the level curves $f(x,y)=\frac{1}{4}$, $f(x,y)=1$, and $f(x,y)=2$ are drawn below. Also labeled are points P, Q, and R. Use them to do the following.



(i) (1 pt) At the point P, is f_x positive, negative, or zero? Circle one answer below.

POSITIVE NEGATIVE ZERO

(ii) (2 pts) At the point Q on the graph, draw a nonzero vector in the direction of greatest instantaneous *decrease* of f at Q.

(iii) (2 pts) At the point R on the graph, draw a nonzero vector v so that the directional derivative of f in the direction of v is zero.

b) (6 points) Let $g(x,y)=\begin{cases}\dfrac{-x^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0);\\[2ex] c & \text{if } (x,y)=(0,0)\end{cases}$

Is there a value of c that makes g continuous at $(0,0)$? If your answer is yes, find c. If your answer is no, justify why there is no value of c that makes g continuous at $(0,0)$.

Space for extra work on problem 2.

3. Part (a) is worth 6 points, while (b) is worth 3 points.

a) Find the linearization $L(x, y)$ of

$$f(x, y) = 12x^{1/3}y^{1/2}$$

at the point $(8, 9)$. Fully simplify your answer.

b) Use your linearization from part (a) to approximate $12 \cdot 7^{1/3} \cdot 11^{1/2}$.

Space for extra work on problem 3.

4. (9 points) Find all critical points of

$$f(x,y) = \frac{2}{3}x^3 - 2xy + y^2 + 2.$$

Classify each critical point as a local maximum, local minimum, or saddle point.

(you don't need to find the values of f at the critical points, you just need to determine whether each critical point gives a local max, local min, or saddle point)

Space for extra work on problem 4.

5. (10 points) Find the point on the curve $x^2 - 2x + y^2 - 6y = 30$ which is furthest from the origin. For that point, find its distance from the origin.

Space for extra work on problem 5.

MATH 2550, FINAL EXAM
SPRING 2019, JANKOWSKI

Name		**Section**	

1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- You exam has 100 points, and you have 170 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will provide loose scrap paper, but the scrap paper will not be collected or graded under any circumstances. Anything that you want to be graded must be written on the exam itself.
- Good luck!

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1. You do not need to show any work on this problem. Parts (a), (c), and (d) are worth two points each, while (b) is worth 3 points.

a) Suppose S_1 and S_2 are planes in $\mathbf{R}^3$ whose intersection includes the point $(1,0,0)$, and suppose that S_1 and S_2 have unit normals $\mathbf{n}_1$ and $\mathbf{n}_2$ (respectively) at $(1,0,0)$.
True or false: $\mathbf{n}_1 \times \mathbf{n}_2$ must be perpendicular to both S_1 and S_2 at $(1,0,0)$.

TRUE FALSE

b) Suppose $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbf{R}^3$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$ (chosen so $0 \le \theta \le \pi$). Which of the following are true? Circle all that apply.

(i) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.

(ii) $\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin(\theta)$.

(iii) If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, then $\mathbf{u} \times \mathbf{v}$ is not the zero vector.

c) Which formula below gives the distance from the point $(1,3,-1)$ to the line $\mathbf{r}(t)$ below?

$$\mathbf{r}(t) = \langle 2-t,\ 3+2t,\ 3t\rangle, \quad -\infty < t < \infty.$$

(I) $\dfrac{|\langle -1,0,-1\rangle \times \langle 2,3,0\rangle|}{|\langle -1,2,3\rangle|}$

(II) $\dfrac{|\langle -1,0,-1\rangle \times \langle -1,2,3\rangle|}{|\langle -1,2,3\rangle|}$

(III) $\dfrac{|\langle 1,3,-1\rangle \times \langle -1,2,3\rangle|}{|\langle -1,2,3\rangle|}$

(IV) $\dfrac{|\langle -1,0,-1\rangle \times \langle -1,2,3\rangle|}{|\langle 1,3,-1\rangle|}$

(V) $\dfrac{|\langle 1,3,-1\rangle \cdot \langle -1,2,3\rangle|}{|\langle -1,2,3\rangle|}$

d) In each case, fill in the blank to identify the quadric surface given by the equation. In order to receive credit, you must be fully specific.

(i) $z^2 = x^2 + y^2 - 1$ ________________________

(ii) $z = 1 - x^2 + y^2$ ________________________

2. You do not need to show your work on this problem. Each part is worth 2 points.

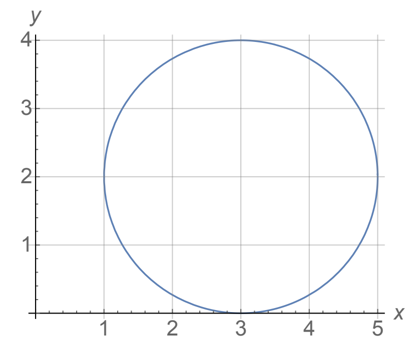
a) Let $\mathbf{r}(t) = \langle \cos(t),\ \sin(t),\ t\rangle$ for $-\infty < t < \infty$. Which formula below gives the arclength of $\mathbf{r}$ from $t = 0$ to $t = 1$?

(I) $\displaystyle\int_0^1 \sqrt{2}\, dt$ (II) $\displaystyle\int_0^1 \sqrt{1+t^2}\, dt$

(III) $\displaystyle\int_0^1 \langle -\sin(t),\ \cos(t),\ 1\rangle\, dt$ (IV) $\displaystyle\int_0^1 \langle \cos(t),\ \sin(t), t\rangle\, dt$

b) Suppose $\mathbf{r}(t)$ is a smooth parametrization of the circle drawn below.

Fill in the blank: the curvature $\kappa(t)$ is __________.



c) If f is a continuous function, which integral below is equal to

$$\int_{-1}^{0}\int_{x^2}^{1} f(x,y)\, dy\, dx + \int_{0}^{2}\int_{x/2}^{1} f(x,y)\, dy\, dx?$$

(I) $\displaystyle\int_0^1\int_{-\sqrt{y}}^{2y} f(x,y)\, dx\, dy$ (II) $\displaystyle\int_0^1\int_{\sqrt{y}}^{y/2} f(x,y)\, dx\, dy$

(III) $\displaystyle\int_0^1\int_{\sqrt{y}}^{2y} f(x,y)\, dx\, dy$ (IV) $\displaystyle\int_{-1}^2\int_{y/2}^{\sqrt{y}} f(x,y)\, dx\, dy$ (V) $\displaystyle\int_{-1}^2\int_{-\sqrt{y}}^{2y} f(x,y)\, dx\, dy$

d) Suppose $g(x, y)$ is a function which is differentiable everywhere. Write the *limit definition* of the partial derivative of g with respect to y at a point (a, b).

$g_y(a, b) =$

3. You do not need to show work or justify your answers in this problem except for part (d). Parts (a) and (c) are two points, while (b) is three points and (d) is four points.

a) Consider the surface $3x^2 - y^2 - z = 1$.
Write one nonzero vector perpendicular to the surface's tangent plane at $(1,1,1)$.

____

b) Each of the following infinitely-differentiable functions have $(0,0)$ as a critical point. Which functions have a *saddle point* at $(0,0)$? Circle all that apply.

(i) $f(x,y) = 1 - x^2 - y^2$

(ii) $g(x,y) = x^3 + y^3$

(iii) $j(x,y) = x^4 + y^4$

c) Let $f(x,y) = x^2 e^y$. What is the linearization $L(x,y)$ of f at $(3,0)$?

(I) $L(x,y) = 6(x-3) + 9y$

(II) $L(x,y) = 9 + 2xe^y(x-3) + x^2 e^y(y-0)$

(III) $0 = 9 + 2xe^y(x-3) + x^2 e^y(y-0)$

(IV) $L(x,y) = 9 + 6(x-3) + 9y$

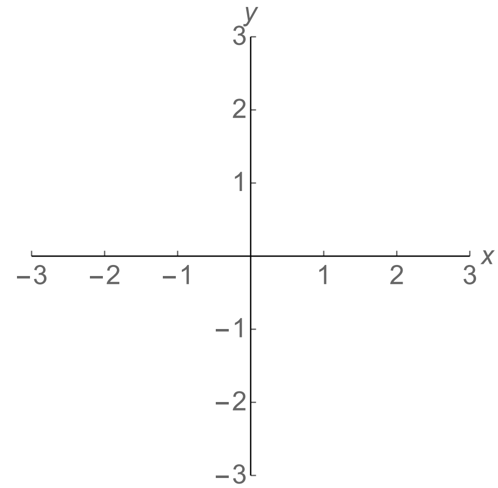
(V) $0 = 9 + 6(x-3) + 9y$

d) Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^3}{x^2 + y^2}$ or show it does not exist. You must fully justify your answer to receive full credit.

4. The rest of the exam is free response. Show your work unless instructed otherwise! A correct answer without appropriate work may receive little or no credit.

a) (6 points) Consider the function $f(x,y) = \sqrt{1-x^2-y}$.

(i) Graph the domain of f on the axes below.



(ii) Fill-in: Is the domain of f open, closed, or neither? ____________________
(no work necessary for (ii))

(iii) Fill-in: Is the domain of f bounded or unbounded? ____________________
(no work necessary for (iii))

b) (4 points) Suppose $T(x,y)$ is a smooth function representing the temperature in degrees Celsius on a plate, where

$$T_x(3,2) = -1 \quad \text{and} \quad T_y(3,2) = 4.$$

Tony the Ant crawls on the plate in a path given by $x = 5-t$ and $y = te^{t-2}$. Find the instantaneous rate of change of temperature on Tony's path when $t = 2$.

5. (10 points) Consider the function $f(x,y) = e^{x^2y}$.

a) Find the directional derivative of f at $(2,0)$ in the direction of $\langle 4, -1\rangle$.

b) Find a unit vector $\mathbf{u}$ so that the directional derivative of f in the direction of $\mathbf{u}$ at $(1,1)$ is zero.

c) Is there a point (a,b) where $\nabla f(a,b) \neq \mathbf{0}$ and the vector $\langle 6, 1\rangle$ points in the direction of greatest increase of f? If so, find such a point (a,b). If not, justify why there is no such point (a,b).

6. (10 points) Use Lagrange multipliers to find all points on the surface $x^2 + zy = 12$ which are closest to the origin. What is this minimum distance?
(For full credit, you must find and test all points satisfying the Lagrange condition.)

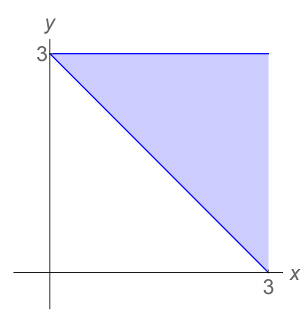
Enter your final answer in the lines below.
The point(s) which are closest to the origin: ________________________________

The minimum distance to the origin: ____________________________

7. (11 points) Parts (a) and (b) are unrelated.

a) Let R be the region of the xy-plane enclosed by the triangle with vertices $(0,0)$, $(1,1)$, and $(2,0)$. Suppose R has density function $\delta(x,y) = x$. Find the mass of R. Fully compute the integral!

b) Write $\displaystyle\iint_S y \, dA$ in polar coordinates, where S is the shaded region below. *Do not evaluate any integrals!*



8. Let D be the region bounded above by the paraboloid $z = 2-x^2-y^2$ and bounded below by the cone $z = \sqrt{x^2+y^2}$.

a) (5 points) Write an integral (or integrals) in rectangular coordinates that would give the volume of D, using the order $dz\ dy\ dx$. *Do not evaluate any integrals!*

b) (6 points) Write an integral (or integrals) in rectangular coordinates that would give the volume of D, using the order $dy\ dx\ dz$. *Do not evaluate any integrals!*

9. **a)** (6 points) Write an integral in spherical coordinates that would give

$$\iiint_D y^2 \, dV,$$

where D is the region bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and bounded below by the surface $z = \dfrac{\sqrt{x^2+y^2}}{\sqrt{3}}$. *Do not evaluate the integral!*

b) (5 points) Write an integral in cylindrical coordinates that would give the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 9$ and bounded below by the paraboloid $z = 2x^2 + 2y^2$. You may use the fact that the sphere and paraboloid intersect when $r = 1.175$. *Do not evaluate the integral!*

10. (9 points) Use the method of substitution to evaluate

$$\iint_R \frac{1}{2+x^2+16y^2}\, dA,$$

where R is the region enclosed by the ellipse $\dfrac{x^2}{16}+y^2=1$. *Fully compute the integral.*