Math 1553
Introduction to Linear Algebra

School of Mathematics
Georgia Institute of Technology

Introduction to Linear Algebra

Motivation and Overview

What is Linear Algebra?

Linear

- having to do with lines/planes/etc.
- For example, $x + y + 3z = 7$, not $\sin, \log, x^2$, etc.

Algebra

- solving equations involving numbers and symbols
- from al-jebr (Arabic), meaning reunion of broken parts
- 9^{th} century Abu Ja'far Muhammad ibn Muso al-Khwarizmi

But these are the easiest kind of equations! I learned how to solve them in 7th grade!

Ah, but engineers need to solve *lots* of equations in *lots* of variables.

$$\begin{array}{rcrcrcrcrcrcr}
3x_1 & + & 4x_2 & + & 10x_3 & + & 19x_4 & - & 2x_5 & - & 3x_6 & = & 141 \\
7x_1 & + & 2x_2 & - & 13x_3 & - & 7x_4 & + & 21x_5 & + & 8x_6 & = & 2567 \\
-x_1 & + & 9x_2 & + & \frac{3}{2}x_3 & + & x_4 & + & 14x_5 & + & 27x_6 & = & 26 \\
\frac{1}{2}x_1 & + & 4x_2 & + & 10x_3 & + & 11x_4 & + & 2x_5 & + & x_6 & = & -15
\end{array}$$

Often, it's enough to know some information about the set of solutions without having to solve the equations at all!

Also, what if one of the coefficients of the x_i is itself a parameter— like an unknown real number t?

In real life, the difficult part is often in recognizing that a problem can be solved using linear algebra in the first place: need *conceptual* understanding.

Large classes of engineering problems, no matter how huge, can be reduced to linear algebra:

$$Ax = b \quad \text{or}$$
$$Ax = \lambda x$$

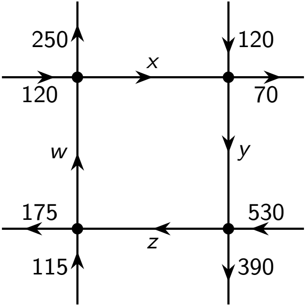
"...and now it's just linear algebra"

Civil Engineering: How much traffic flows through the four labeled segments?

$\rightsquigarrow$ system of linear equations:

$$\begin{aligned} w + 120 &= x + 250 \\ x + 120 &= y + 70 \\ y + 530 &= z + 390 \\ z + 115 &= w + 175 \end{aligned}$$

Traffic flow (cars/hr)



Chemistry: Balancing reaction equations

$$\underline{\ x\ }\ C_2H_6 + \underline{\ y\ }\ O_2 \to \underline{\ z\ }\ CO_2 + \underline{\ w\ }\ H_2O$$

$\rightsquigarrow$ system of linear equations, one equation for each element.

$$\begin{aligned} 2x &= z \\ 6x &= 2w \\ 2y &= 2z + w \end{aligned}$$

Biology: In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population in 2016 (in terms of the number of first, second, and third year rabbits), then what is the population in 2017?

$\rightsquigarrow$ system of linear equations:

$$\begin{aligned} 6y_{2016} + 8z_{2016} &= x_{2017} \\ \tfrac{1}{2}x_{2016} \qquad\qquad &= y_{2017} \\ \tfrac{1}{2}y_{2016} \qquad\qquad &= z_{2017} \end{aligned}$$

Question

Does the rabbit population have an asymptotic behavior? Is this even a linear algebra question? Yes, it is!

Geometry and Astronomy: Find the equation of a circle passing through 3 given points, say $(1,0)$, $(0,1)$, and $(1,1)$. The general form of a circle is $a(x^2+y^2)+bx+cy+d=0$.

$\rightsquigarrow$ system of linear equations:

$$\begin{aligned} a + b \phantom{{}+c} + d &= 0 \\ a \phantom{{}+b} + c + d &= 0 \\ 2a + b + c + d &= 0 \end{aligned}$$

Very similar to: compute the orbit of a planet:

$$ax^2+by^2+cxy+dx+ey+f=0$$

Google: "The 25 billion dollar eigenvector." Each web page has some importance, which it shares via outgoing links to other pages
$\rightsquigarrow$ system of linear equations (in gazillions of variables).

Larry Page flies around in a private 747 partly because he paid attention in his linear algebra class.

Observables like position and momentum $\rightsquigarrow$ linear operators

The "wave function" ψ_t (where t is the time parameter), in some number of variables: satisfies Schrodinger's equation for the system's Hamiltonian.

How to do computations:
The value of an observable A at time t is the inner product (like a dot product) of ψ_t with $A\psi_t$. This is actually given by an integral!

In this way, quantum mechanics relates to linear algebra.

Unfortunately: this part is beyond the scope of Math 1553.

Overview of the Course

- Solve the matrix equation $Ax = b$
  - Solve systems of linear equations using matrices, row reduction, and inverses.
  - Solve systems of linear equations with varying parameters using parametric forms for solutions, the geometry of linear transformations, the characterizations of invertible matrices, and determinants.
- Solve the matrix equation $Ax = \lambda x$
  - Solve eigenvalue problems through the use of the characteristic polynomial.
  - Understand the dynamics of a linear transformation via the computation of eigenvalues, eigenvectors, and diagonalization.
- Almost solve the equation $Ax = b$
  - Find best-fit solutions to systems of linear equations that have no actual solution using least squares approximations.

What to Expect This Semester

Your previous math courses probably focused on how to do (sometimes rather involved) computations.

- Compute the derivative of $\sin(\log x)\cos(e^x)$.
- Compute $\int_0^1 (1 - \cos(x))\,dx$.

This is important, **but** Wolfram Alpha can do all these problems better than any of us can. Nobody is going to hire us to do something a computer can do better.

> If a computer can do the problem better than we can, then it's just an algorithm: this is not real problem solving.

So what are we going to do?

- About half the material focuses on how to do linear algebra computations—that is still important.
- The other half is on *conceptual* understanding of linear algebra. This is much more subtle: it's about figuring out *what question* to ask the computer, or whether you actually need to do any computations at all.

Resources

See our lecture's webpage (linked from T-Square and my personal webpage) for the following.

- Calendar: lists what will happen on which day, links to daily slides, quizzes, practice exams, solutions, etc.
- A syllabus which provides information such as:
  - Course administration: the names of your TAs, their office hours, your recitation location, etc., are on the syllabus.
  - Course organization: grading policies, details about homework and exams, etc.
- A link to the master website for Math 1553, which includes advice and help: how to succeed in this course, helpful links, and more.

T-Square: your grades, link to WeBWorK.

Piazza: forum for asking questions and discussing mathematics. We will be using Piazza for the class participation grade, starting the second week of class.