Section 1.4

The Matrix Equation $Ax = b$

Matrix × Vector

the first number is
the number of rows

the second number is
the number of columns

Let A be an $\underset{\uparrow}{m} \times \underset{\uparrow}{n}$ matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \text{with columns } v_1, v_2, \ldots, v_n$$

Definition

The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

this means the equality
is a *definition*

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \boxed{\overset{\text{def}}{=}} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

these must be equal

The output is a vector in $\mathbf{R}^m$.

Note that the number of columns of A has to equal the number of rows of x.

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Question

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

Answer: Let A be the matrix with colums v_1, v_2, v_3, and let x be the vector with entries $2, 3, -4$. Then

$$Ax = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 2v_1 + 3v_2 - 4v_3,$$

so the vector equation is equivalent to the matrix equation

$$Ax = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}.$$

Let $v_1, v_2, \ldots, v_n$, and b be vectors in $\mathbf{R}^m$. Consider the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

It is equivalent to the **matrix equation**

$$Ax = b$$

where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, if A is any $m \times n$ matrix, then

$$Ax = b \qquad \begin{array}{c}\text{is equivalent to the} \\ \text{vector equation}\end{array} \qquad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$$

where $v_1, \ldots, v_n$ are the columns of A, and $x_1, \ldots, x_n$ are the entries of x.

Linear Systems, Vector Equations, Matrix Equations, ...

We now have *four* equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. As an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$

3. As a vector equation ($x_1 v_1 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation ($Ax = b$):

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

We will move back and forth freely between these over and over again, for the rest of the semester. Get comfortable with them now!

In particular, *all four have the same solution set*.

Definition

A **row vector** is a matrix with one row. The product of a row vector of length n and a (column) vector of length n is

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \overset{\text{def}}{=} a_1x_1 + \cdots + a_nx_n.$$

This is a scalar.

If A is an $m \times n$ matrix with rows $r_1, r_2, \ldots, r_m$, and x is a vector in $\mathbf{R}^n$, then

$$Ax = \begin{pmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — r_m — \end{pmatrix} x = \begin{pmatrix} r_1x \\ r_2x \\ \vdots \\ r_mx \end{pmatrix}$$

This is a vector in $\mathbf{R}^m$ (again).

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \left(\begin{smallmatrix} 4 & 5 & 6 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 7 & 8 & 9 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right) \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 \\ 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Note this is the same as before:

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\ 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Now you have *two* ways of computing Ax.

In the second, you calculate Ax one entry at a time.

The second way is usually the most convenient, but we'll use both.

Spans and Solutions to Equations

Let A be a matrix with columns $v_1, v_2, \ldots, v_n$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

Very Important Fact That Will Appear on Every Midterm and the Final

$Ax = b$ has a solution

$$\iff \text{there exist } x_1, \ldots, x_n \text{ such that } A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

"if and only if"

$$\iff \text{there exist } x_1, \ldots, x_n \text{ such that } x_1 v_1 + \cdots + x_n v_n = b$$

$$\iff b \text{ is a linear combination of } v_1, \ldots, v_n$$

$$\iff b \text{ is in the span of the columns of } A.$$

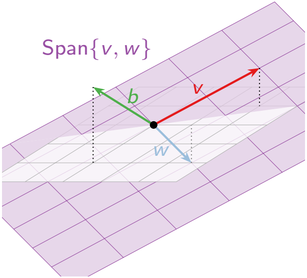
The last condition is geometric.

Spans and Solutions to Equations
Example

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

[interactive]



Columns of A:

$$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Output vector:

$$b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A? It sure doesn't look like it.

Conclusion: $Ax = b$ is *inconsistent*.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's check by solving the matrix equation using row reduction.

The first step is to put the system into an augmented matrix.

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

The last equation is $0 = 1$, so the system is *inconsistent*.

In other words, the matrix equation

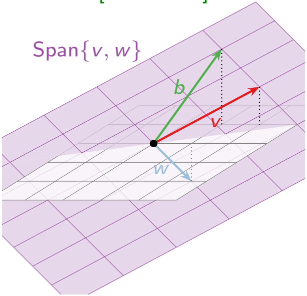
$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

has no solution, as the picture shows.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

[interactive]



Columns of A:

$$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Solution vector:

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A? It looks like it: in fact,

$$b = 1v + (-1)w \implies x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's do this systematically using row reduction.

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

This gives us

$$x = 1 \qquad y = -1.$$

This is consistent with the picture on the previous slide:

$$1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad \text{or} \qquad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Poll

Poll

True or false: (can be done by eyeballing equation)

The matrix equation $\begin{pmatrix} 1 & -3 & 0 \\ 0 & 2 & 3 \\ 0 & -1 & 6 \end{pmatrix} \mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbf{R}^3$.

A. True

B. False

When Solutions Always Exist

Here are criteria for a linear system to *always* have a solution.

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution *for all* b in $\mathbf{R}^m$.
2. The span of the columns of A is all of $\mathbf{R}^m$.
3. A has a pivot in each row.

recall that this means that for given A, either they're all true, or they're all false

Why is (1) the same as (2)? This was the $\boxed{\text{Very Important}}$ box from before.

Why is (1) the same as (3)? If A has a pivot in each row then its reduced row echelon form looks like this:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix} \quad \text{and } (\,A \mid b\,) \text{ reduces to this: } \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & \star \\ 0 & 1 & \star & 0 & \star & \star \\ 0 & 0 & 0 & 1 & \star & \star \end{array}\right).$$

There's no b that makes it inconsistent, so there's always a solution. If A doesn't have a pivot in each row, then its reduced form looks like this:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and this can be made inconsistent: } \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & 0 \\ 0 & 1 & \star & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{array}\right).$$

When Solutions Always Exist
Continued

Theorem
Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution *for all* b in $\mathbf{R}^m$.
2. The span of the columns of A is all of $\mathbf{R}^m$.
3. A has a pivot in each row.

In the following demos, the red region is the span of the columns of A. This is the same as the set of all b such that $Ax = b$ has a solution.

[example where the criteria are satisfied]

[example where the criteria are not satisfied]

Properties of the Matrix–Vector Product

Let c be a scalar, u, v be vectors, and A a matrix.

- $A(u + v) = Au + Av$
- $A(cv) = cAv$

See Lay, §1.4, Theorem 5.

For instance, $A(3u - 7v) = 3Au - 7Av$.

Consequence: If u and v are solutions to $Ax = 0$, then so is every vector in Span$\{u, v\}$. Why?

$$\begin{cases} Au = 0 \\ Av = 0 \end{cases} \implies A(xu + yv) = xAu + yAv = x0 + y0 = 0.$$

(Here 0 means the zero vector.)

Important

The set of solutions to $Ax = 0$ is a span.