Section 2.3

Characterization of Invertible Matrices

Invertible Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists another transformation $U\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

for all x in $\mathbf{R}^n$. In this case we say U is the **inverse** of T, and we write $U = T^{-1}$.

In other words, $T(U(x)) = x$, so T "undoes" U, and likewise U "undoes" T.

Fact

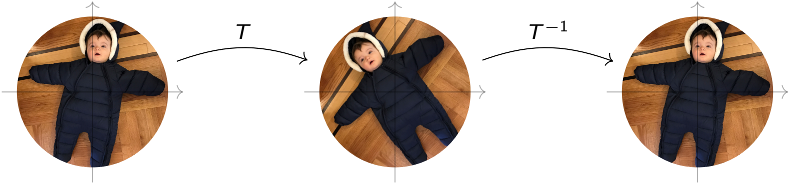
A transformation T is invertible if and only if it is both one-to-one and onto.

If T is one-to-one and onto, this means for every y in $\mathbf{R}^n$, there is a unique x in $\mathbf{R}^n$ such that $T(x) = y$. Then $T^{-1}(y) = x$.

Invertible Transformations

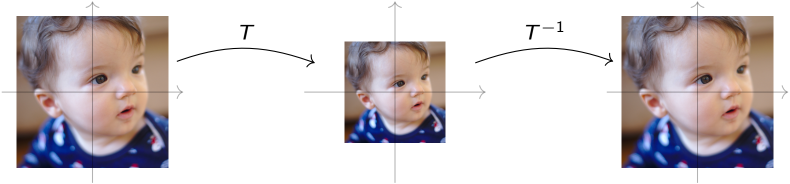
Examples

Let $T =$ counterclockwise rotation in the plane by 45°. What is T^{-1}?



T^{-1} is *clockwise* rotation by 45°. [interactive: $T^{-1} \circ T$] [interactive: $T \circ T^{-1}$]

Let $T =$ shrinking by a factor of $2/3$ in the plane. What is T^{-1}?



T^{-1} is *stretching* by $3/2$. [interactive: $T^{-1} \circ T$] [interactive: $T \circ T^{-1}$]

Let $T =$ projection onto the x-axis. What is T^{-1}? It is not invertible: you can't undo it.

Invertible Linear Transformations

If $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is an invertible *linear* transformation with matrix A, then what is the matrix for T^{-1}?

Let B be the matrix for T^{-1}. We know $T \circ T^{-1}$ has matrix AB, so for all x,

$$ABx = T \circ T^{-1}(x) = x.$$

Hence $AB = I_n$, so $B = A^{-1}$.

Fact

If T is an invertible linear transformation with matrix A, then T^{-1} is an invertible linear transformation with matrix A^{-1}.

Let T = counterclockwise rotation in the plane by 45°. Its matrix is

$$A = \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then T^{-1} = counterclockwise rotation by -45°. Its matrix is

$$B = \begin{pmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Check: $$AB = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ✓$$

Let T = shrinking by a factor of $2/3$ in the plane. Its matrix is

$$A = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/3 \end{pmatrix}$$

Then T^{-1} = stretching by $3/2$. Its matrix is

$$B = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$

Check: $$AB = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/3 \end{pmatrix} \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ✓$$

The Invertible Matrix Theorem

A.K.A. The Really Big Theorem of Math 1553

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.

you really have to know these

There are two kinds of *square* matrices:

1. invertible (non-singular), and
2. non-invertible (singular).

For invertible matrices, all statements of the Invertible Matrix Theorem are true.

For non-invertible matrices, all statements of the Invertible Matrix Theorem are false.

Strong recommendation: If you want to understand invertible matrices, go through all of the conditions of the IMT and try to figure out on your own (or at least with help from the book) why they're all equivalent.

You know enough at this point to be able to reduce all of the statements to assertions about the pivots of a square matrix.