Chapter 3

Determinants (3.1, 3.2, and some of 3.3)

Orientation

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
  We've said most of what we'll say about this topic now.
- Solve the matrix equation $Ax = \lambda x$ (eigenvalue problem)
  We are now aiming at this.
- Almost solve the equation $Ax = b$
  This will happen later.

The next topic is *determinants*.

This is a completely *magical* function that takes a square matrix and gives you a number.

It is a very complicated function—the formula for the determinant of a 10×10 matrix has $3,628,800$ summands—so instead of writing down the formula, we'll give other ways to compute it.

Today is mostly about the *theory* of the determinant; in the next lecture we will focus on *computation*.

A Definition of Determinant

We will define determinants in terms of row operations. This immediately tells us how to compute them.

determinants are only for square matrices!

Definition

The **determinant** of a $n \times n$ square matrix A is a number $\det(A)$ such that:

1. If you do a row replacement on A, the determinant doesn't change.
2. If you scale a row of A by c, the determinant is multiplied by c.
3. If you do a row swap on A, the determinant is multiplied by -1.
4. $\det(I_n) = 1$.

Example:

$$\begin{pmatrix}2&1\\1&4\end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix}1&4\\2&1\end{pmatrix} \qquad \det = 7$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix}1&4\\0&-7\end{pmatrix} \qquad \det = -7$$

$$\xrightarrow{R_2 = R_2 \div -7} \begin{pmatrix}1&4\\0&1\end{pmatrix} \qquad \det = 1$$

$$\xrightarrow{R_1 = R_1 - 4R_2} \begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad \det = 1$$

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3. If you do a row swap on A, the determinant is multiplied by -1.
4. $\det(I_n) = 1$.

> This is a *definition* because it tells you how to compute the determinant: row reduce!

It's not at all obvious that you get the same determinant if you row reduce in two different ways, but this is *magically* true!

Special Cases

Special Case 1

If A has a zero row, then $\det(A) = 0$.

Why?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 = -R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}$$

The determinant of the second matrix is negative the determinant of the first (property 3), so

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} = -\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}.$$

This implies the determinant is zero.

Special Cases

Special Case 2

If A is upper-triangular, then the determinant is the product of the diagonal entries:

$$\det\begin{pmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{pmatrix} = abc.$$

Upper-triangular means the only nonzero entries are on or above the diagonal.

Why?

- If one of the diagonal entries is zero, then the matrix has fewer than n pivots, so the RREF has a row of zeros. (Row operations don't change whether the determinant is zero.)
- Otherwise,

$$\begin{pmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{pmatrix} \xrightarrow[\quad]{\text{scale by } a^{-1}, b^{-1}, c^{-1}} \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\quad]{\text{row replacements}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det = abc \quad \longleftarrow \quad \det = 1 \quad \longleftarrow \quad \det = 1$$

Theorem

Let A be a square matrix. Suppose you do some number of row operations on A to get a matrix B in row echelon form. Then

$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})},$$

where r is the number of row swaps.

Why? Since B is in REF, it is upper-triangular, so its determinant is the product of its diagonal entries. You changed the determinant by $(-1)^r$ and the product of the scaling factors when going from A to B.

Remark

This is generally the fastest way to compute a determinant of a large matrix, either by hand or by computer.

Row reduction is $O(n^3)$; cofactor expansion (next time) is $O(n!) \sim O(n^n\sqrt{n})$.

This is important in real life, when you're usually working with matrices with a gazillion columns.

Computing Determinants

Example

$$\begin{pmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} \xrightarrow[\rightsquigarrow]{R_1 \longleftrightarrow R_2} \begin{pmatrix} 2 & 4 & 6 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix} \quad r = 1$$

$$\xrightarrow[\rightsquigarrow]{R_1 = R_1 \div 2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix} \quad \begin{array}{l} r = 1 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\xrightarrow[\rightsquigarrow]{R_3 = R_3 - 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 1 & -10 \end{pmatrix} \quad \begin{array}{l} r = 1 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\xrightarrow[\rightsquigarrow]{R_2 \longleftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & -7 & -4 \end{pmatrix} \quad \begin{array}{l} r = 2 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\xrightarrow[\rightsquigarrow]{R_3 = R_3 + 7R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & -74 \end{pmatrix} \quad \begin{array}{l} r = 2 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\Longrightarrow \det \begin{pmatrix} 0 & -7 & -4 \\ 1 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} = (-1)^2 \frac{1 \cdot 1 \cdot -74}{1/2} = -148.$$

Let's compute the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a general 2×2 matrix.

- If $a = 0$, then
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = -\det \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = -bc.$$

- Otherwise,
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot \det \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} = a \cdot \det \begin{pmatrix} 1 & b/a \\ 0 & d - c \cdot b/a \end{pmatrix}$$
$$= a \cdot 1 \cdot (d - bc/a) = ad - bc.$$

In both cases, the determinant *magically* turns out to be
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Poll

Poll

Suppose that A is a 4×4 matrix satisfying

$$Ae_1 = e_2 \quad Ae_2 = e_3 \quad Ae_3 = e_4 \quad Ae_4 = e_1.$$

What is $\det(A)$?

A. -1 B. 0 C. 1

These equations tell us the columns of A:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

You need 3 row swaps to transform this to the identity matrix.
So $\det(A) = (-1)^3 = -1$.

Determinants and Invertibility

Theorem

A square matrix A is invertible if and only if $\det(A)$ is nonzero.

Why?

- If A is invertible, then its reduced row echelon form is the identity matrix, which has determinant equal to 1.
- If A is not invertible, then its reduced row echelon form has a zero row, hence has zero determinant.

Determinants and Products

Theorem

If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

Why? If B is invertible, we can define

$$f(A) = \frac{\det(AB)}{\det(B)}.$$

Note $f(I_n) = \det(I_nB)/\det(B) = 1$. Check that f satisfies the same properties as det with respect to row operations. So

$$\det(A) = f(A) = \frac{\det(AB)}{\det(B)} \implies \det(AB) = \det(A)\det(B).$$

What about if B is not invertible?

Theorem

If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Why? $I_n = AB \implies 1 = \det(I_n) = \det(AB) = \det(A)\det(B)$.

Determinants and Transposes

Theorem

If A is a square matrix, then

$$\det(A) = \det(A^T),$$

where A^T is the transpose of A.

Example: $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

As a consequence, det behaves the same way with respect to *column* operations as row operations.

Corollary an immediate consequence of a theorem

If A has a zero column, then $\det(A) = 0$.

Corollary

The determinant of a *lower*-triangular matrix is the product of the diagonal entries.

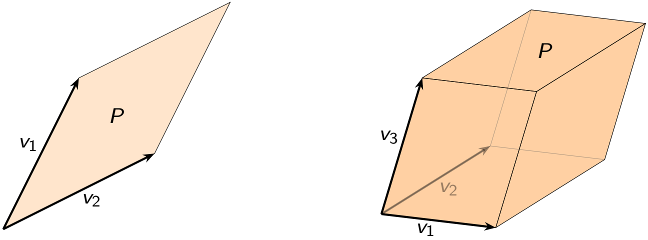
(The transpose of a lower-triangular matrix is upper-triangular.)

Determinants and Volumes

Now we discuss a completely different description of (the absolute value of) the determinant, in terms of volumes.

This is a crucial component of the change-of-variables formula in multivariable calculus.

The columns $v_1, v_2, \ldots, v_n$ of an $n \times n$ matrix A give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.



Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

Determinants and Volumes

Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

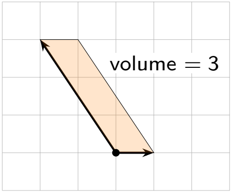
Sanity check: the volume of P is zero $\iff$ the columns are *linearly dependent* (P is "flat") $\iff$ the matrix A is not invertible.

Why is the theorem true? First you have to defined a "signed" volume, i.e. to figure out when a volume should be negative.

Then you have to check that the volume behaves the same way under row operations as the determinant does.

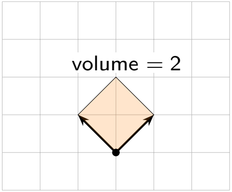
Note that the volume of the unit cube (the parallelepiped defined by the identity matrix) is 1.

$$\det\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$

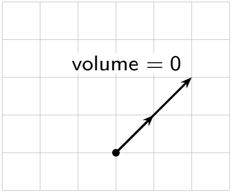


$$\det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -2$$

(Should the volume really be -2?)



$$\det\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 0$$



Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

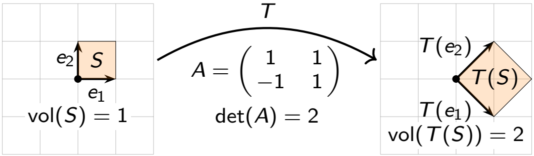
This is even true for curvy shapes, in the following sense.

Theorem

Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

If S is the unit cube, then $T(S)$ is the parallelepiped defined by the columns of A, since the columns of A are $T(e_1), T(e_2), \ldots, T(e_n)$. In this case, the second theorem is the same as the first.



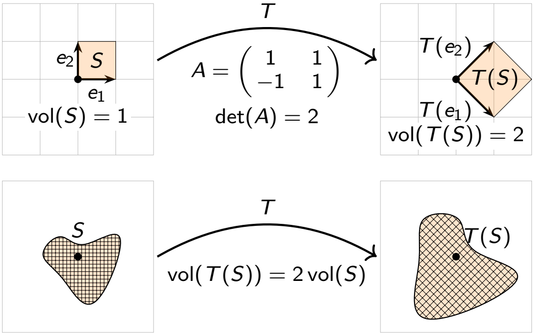
Determinants and Volumes

Theorem

Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

For curvy shapes, you break S up into a bunch of tiny cubes. Each one is scaled by $|\det(A)|$; then you use *calculus* to reduce to the previous situation!



Theorem

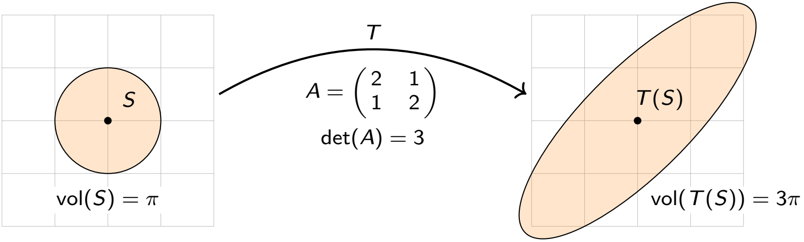
Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

Example: Let S be the unit disk in $\mathbf{R}^2$, and let $T(x) = Ax$ for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Note that $\det(A) = 3$.



Magical Properties of the Determinant

you really have to know these

1. There is one and only one function $\det\colon \{\text{square matrices}\} \to \mathbf{R}$ satisfying the properties (1)–(4) on the second slide.
2. A is invertible if and only if $\det(A) \neq 0$.
3. If we row reduce A to row echelon form B using r swaps, then
$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})}.$$
4. $\det(AB) = \det(A)\det(B) \quad \text{and} \quad \det(A^{-1}) = \det(A)^{-1}$.
5. $\det(A) = \det(A^T)$.
6. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.
7. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)

Orientation

Last time: we learned. . .

- . . . the definition of the determinant.
- . . . to compute the determinant using row reduction.
- . . . all sorts of *magical* properties of the determinant, like
  - $\det(AB) = \det(A)\det(B)$
  - the determinant computes volumes
  - nonzero determinants characterize invertability
  - etc.

Today: we will learn. . .

- Special formulas for 2×2 and 3×3 matrices.
- How to compute determinants using *cofactor expansions*.
- How to compute inverses using determinants.

Determinants of 2×2 Matrices

Reminder

We already have a formula in the 2×2 case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Determinants of 3×3 Matrices

Here's the formula:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

How on earth do you remember this? Draw a bigger matrix, repeating the first two columns to the right:

$$+ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$

Then add the products of the downward diagonals, and subtract the product of the upward diagonals. For example,

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = \begin{vmatrix} 5 & 1 & 0 & 5 & 1 \\ -1 & 3 & 2 & -1 & 3 \\ 4 & 0 & -1 & 4 & 0 \end{vmatrix} = -15 + 8 + 0 - 0 - 0 - 1 = -8$$

Cofactor Expansions

When $n \geq 4$, the determinant isn't just a sum of products of diagonals. The formula is *recursive*: you compute a larger determinant in terms of smaller ones.

First some notation. Let A be an $n \times n$ matrix.

$$\begin{aligned} A_{ij} &= ij\text{th } \textbf{minor} \text{ of } A \\ &= (n-1) \times (n-1) \text{ matrix you get by deleting the } i\text{th row and } j\text{th column} \\ C_{ij} &= (-1)^{i+j} \det A_{ij} \\ &= ij\text{th } \textbf{cofactor} \text{ of } A \end{aligned}$$

The signs of the cofactors follow a checkerboard pattern:

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix} \qquad \pm \text{ in the } ij \text{ entry is the sign of } C_{ij}$$

Theorem

The determinant of an $n \times n$ matrix A is

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

This formula is called **cofactor expansion along the first row**.

This is the beginning of the recursion.

$$\det(\, a_{11} \,) = a_{11}.$$

Cofactor Expansions
2×2 Matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The minors are:

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} \end{pmatrix} \qquad A_{12} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{21} \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{12} \end{pmatrix} \qquad A_{22} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \end{pmatrix}$$

The cofactors are

$$C_{11} = +\det A_{11} = a_{22} \qquad C_{12} = -\det A_{12} = -a_{21}$$

$$C_{21} = -\det A_{21} = -a_{12} \qquad C_{22} = +\det A_{22} = a_{11}$$

The determinant is

$$\det A = a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} - a_{12}a_{21}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The top row minors and cofactors are:

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \qquad C_{11} = +\det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \qquad C_{12} = -\det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad C_{13} = +\det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

The determinant is *magically* the same formula as before:

$$\begin{aligned} \det A &= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} \\ &= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \end{aligned}$$

Cofactor Expansions

Example

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = 5\cdot\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & -3 & 2 \\ 4 & 0 & -1 \end{pmatrix} - 1\cdot\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$+\,0\cdot\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$= 5\cdot\det\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} - 1\cdot\det\begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix} + 0\cdot\det\begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix}$$

$$= 5\cdot(-3-0) - 1\cdot(1-8)$$

$$= -15 + 7 = -8$$

$2n-1$ More Formulas for the Determinant

Recall: the formula

$$\det(A) = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

is called **cofactor expansion along the first row.** Actually, you can expand cofactors along any row or column you like!

Magical Theorem

The determinant of an $n \times n$ matrix A is

$$\det A = \sum_{j=1}^{n} a_{ij}C_{ij} \quad \text{for any fixed } i$$

$$\det A = \sum_{i=1}^{n} a_{ij}C_{ij} \quad \text{for any fixed } j$$

These formulas are called **cofactor expansion along the ith row**, respectively, **jth column**.

In particular, you get the *same answer* whichever row or column you choose.

Try this with a row or a column with a lot of zeros.

$$A = \begin{pmatrix} 2 & 1 & \boxed{0} \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

It looks easiest to expand along the third column:

$$\det A = 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} - 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 2 & 1 & \cancel{0} \\ 1 & 1 & \cancel{0} \\ \cancel{5} & \cancel{9} & \cancel{1} \end{pmatrix}$$

$$= \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 2 - 1 = 1$$

Poll

Poll

$$\det \begin{pmatrix} 0 & 7 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} = ?$$

A. -84 **B.** -28 **C.** -7 **D.** 0 **E.** 7 **F.** 28 **G.** 84

Repeatedly expanding along the first column repeatedly, you get:

$$\det \begin{pmatrix} 0 & 7 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} = -1 \cdot \det \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$$= (-1) \cdot 7 \cdot \det \begin{pmatrix} 0 & 0 & 3 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = (-1) \cdot 7 \cdot 2 \cdot \det \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$$

$$= (-1) \cdot 7 \cdot 2 \cdot 6 = -84.$$

A Formula for the Inverse

For fun—from §3.3

For 2×2 matrices we had a nice formula for the inverse:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix}.$$

Theorem

This last formula works for any $n \times n$ invertible matrix A:

$(3, 1)$ entry

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn} \end{pmatrix} = \frac{1}{\det A} \left(C_{ij} \right)^T$$

Note that the cofactors are "transposed": the (i, j) entry of the matrix is C_{ji}.

The proof uses Cramer's rule.

Compute A^{-1}, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The minors are:

$$A_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$
$$A_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{23} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$A_{31} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad A_{32} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A_{33} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The cofactors are (don't forget to multiply by $(-1)^{i+j}$):

$$C_{11} = -1 \qquad C_{12} = 1 \qquad C_{13} = -1$$
$$C_{21} = 1 \qquad C_{22} = -1 \qquad C_{23} = -1$$
$$C_{31} = -1 \qquad C_{32} = -1 \qquad C_{33} = 1$$

The determinant is (expanding along the first row):

$$\det A = 1 \cdot C_{11} + 0 \cdot C_{12} + 1 \cdot C_{13} = -2$$

A Formula for the Inverse

Example, continued

Compute A^{-1}, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The inverse is

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot -\frac{1}{2} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad \checkmark$$

Summary

We have several ways to compute the determinant of a matrix.

- Special formulas for 2×2 and 3×3 matrices.

  These work great for small matrices.

- Cofactor expansion.

  This is perfect when there is a row or column with a lot of zeros.

- Row reduction.

  This is the way to go when you have a big matrix which doesn't have a row or column with a lot of zeros.

- Any combination of the above.

  Cofactor expansion is recursive, but you don't have to use cofactor expansion to compute the determinants of the minors!