Section 5.5

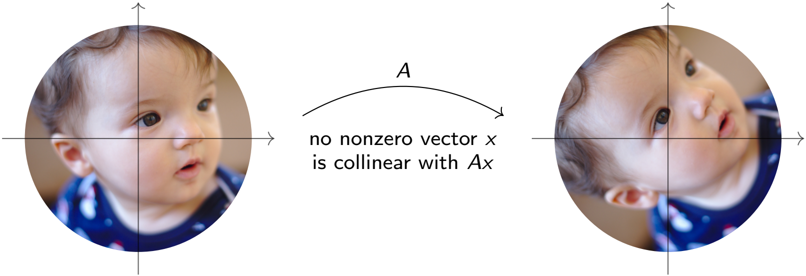
Complex Eigenvalues

A Matrix with No Eigenvectors

In recitation you discussed the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix has no eigenvectors, as you can see geometrically:



or algebraically:

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \sqrt{2}\,\lambda + 1 \implies \lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$

It makes us sad that -1 has no square root. If it did, then $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$.

Mathematician's solution: we're just not using enough numbers! We're going to declare by *fiat* that there exists a square root of -1.

Definition

The number i is defined such that $i^2 = -1$.

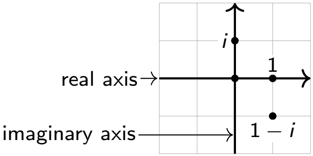
Once we have i, we have to allow numbers like $a + bi$ for real numbers a, b.

Definition

A *complex number* is a number of the form $a + bi$ for a, b in $\mathbf{R}$. The set of all complex numbers is denoted $\mathbf{C}$.

Note $\mathbf{R}$ is contained in $\mathbf{C}$: they're the numbers $a + 0i$.

We can identify $\mathbf{C}$ with $\mathbf{R}^2$ by $a + bi \longleftrightarrow \binom{a}{b}$. So when we draw a picture of $\mathbf{C}$, we draw the plane:



Why This Is Not A Weird Thing To Do

An anachronistic historical aside

In the beginning, people only used counting numbers for, well, counting things: $1, 2, 3, 4, 5, \ldots$. Then someone (Persian mathematician Muḥammad ibn Mūsā al-Khwārizmī, 825) had the ridiculous idea that there should be a number 0 that represents an absence of quantity. This blew everyone's mind.

Then it occurred to someone (Chinese mathematician Liu Hui, c. 3rd century) that there should be *negative* numbers to represent a deficit in quantity. That seemed reasonable, until people realized that $10 + (-3)$ would have to equal 7. This is when people started saying, "bah, math is just too hard for me."

At this point it was inconvenient that you couldn't divide 2 by 3. Thus someone (Indian mathematician Aryabhatta, c. 5th century) invented fractions (rational numbers) to represent fractional quantities. These proved very popular. The Pythagoreans developed a whole belief system around the notion that any quantity worth considering could be broken down into whole numbers in this way.

Then the Pythagoreans (c. 6th century BCE) discovered that the hypotenuse of an isosceles right triangle with side length 1 (i.e. $\sqrt{2}$) is not a fraction. This caused a serious existential crisis and led to at least one death by drowning. The real number $\sqrt{2}$ was thus invented to solve the equation $x^2 - 2 = 0$.

So what's so strange about inventing a number i to solve the equation $x^2 + 1 = 0$?

Operations on Complex Numbers

Addition: $(2-3i)+(-1+i)=1-2i$.

Multiplication: $(2-3i)(-1+i)=2(-1)+2i+3i-3i^2=-2+5i+3=1+5i$.

Complex conjugation: $\overline{a+bi}=a-bi$ is the **complex conjugate** of $a+bi$.
Check: $\overline{z+w}=\overline{z}+\overline{w}$ and $\overline{zw}=\overline{z}\cdot\overline{w}$.

Absolute value: $|a+bi|=\sqrt{a^2+b^2}$. This is a *real* number.
Note: $(a+bi)(\overline{a+bi})=(a+bi)(a-bi)=a^2-(bi)^2=a^2+b^2$. So $|z|=\sqrt{z\overline{z}}$.
Check: $|zw|=|z|\cdot|w|$.

Division by a nonzero real number: $\dfrac{a+bi}{c}=\dfrac{a}{c}+\dfrac{b}{c}i$.

Division by a nonzero complex number: $\dfrac{z}{w}=\dfrac{z\overline{w}}{w\overline{w}}=\dfrac{z\overline{w}}{|w|^2}$.

Example:

$$\frac{1+i}{1-i}=\frac{(1+i)^2}{1^2+(-1)^2}=\frac{1+2i+i^2}{2}=i.$$

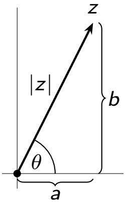
Real and imaginary part: $\mathrm{Re}(a+bi)=a \qquad \mathrm{Im}(a+bi)=b$.

Polar Coordinates for Complex Numbers

Any complex number $z = a + bi$ has the polar coordinates

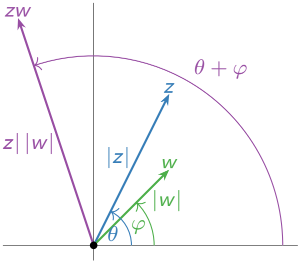
$$z = |z|(\cos\theta + i\sin\theta).$$

The angle θ is called the **argument** of z, and is denoted $\theta = \arg(z)$. Note $\arg(\overline{z}) = -\arg(z)$.



When you multiply complex numbers, you multiply the absolute values and add the arguments:

$$|zw| = |z|\,|w| \qquad \arg(zw) = \arg(z) + \arg(w).$$



Poll

Does every square matrix have an eigenvalue, if we include complex eigenvalues?

Yes. Every polynomial can be factored completely if we include complex numbers, so the characteristic polynomial will have at least one root!

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$f(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Important

If f is a polynomial with *real* coefficients, and if λ is a root of f, then so is $\overline{\lambda}$:

$$\begin{aligned} 0 = \overline{f(\lambda)} &= \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0} \\ &= \overline{\lambda}^n + a_{n-1}\overline{\lambda}^{n-1} + \cdots + a_1\overline{\lambda} + a_0 = f\left(\overline{\lambda}\right). \end{aligned}$$

Therefore complex roots of real polynomials come in *conjugate pairs*.

Degree 2: The quadratic formula gives you the (real or complex) roots of any degree-2 polynomial:

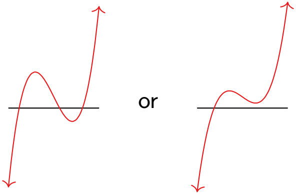
$$f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

For instance, if $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$ then

$$\lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}}{2}(1 \pm i) = \frac{1 \pm i}{\sqrt{2}}.$$

Note the roots are complex conjugates if b, c are real.

Degree 3: A real cubic polynomial has either three real roots, or one real root and a conjugate pair of complex roots. The graph looks like:



respectively.

Example: let $f(\lambda) = 5\lambda^3 - 18\lambda^2 + 21\lambda - 10$.

Since $f(2) = 0$, we can do polynomial long division by $\lambda - 2$: we get $f(\lambda) = (\lambda - 2)\left(5\lambda^2 - 8\lambda + 5\right)$. Using the quadratic formula, the second polynomial has a root when

$$\lambda = \frac{8 \pm \sqrt{64 - 100}}{10} = \frac{4}{5} \pm \frac{\sqrt{-36}}{10} = \frac{4 \pm 3i}{5}.$$

Therefore,

$$f(\lambda) = 5(\lambda - 2)\left(\lambda - \frac{4+3i}{5}\right)\left(\lambda - \frac{4-3i}{5}\right).$$

A Matrix *with* an Eigenvector

Every matrix is guaranteed to have *complex* eigenvalues and eigenvectors. Using rotation by $\pi/4$ from before:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{has eigenvalues} \quad \lambda = \frac{1 \pm i}{\sqrt{2}}.$$

Let's compute an eigenvector for $\lambda = (1+i)/\sqrt{2}$:

$$A - \lambda I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

The second row is i times the first, so we row reduce:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{divide by } -i/\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, so an eigenvector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

A similar computation shows that an eigenvector for $\lambda = (1-i)/\sqrt{2}$ is $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

So is $i\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ (you can scale by *complex* numbers).

A Trick for Computing Eigenvectors of 2×2 Matrices

Very useful for complex eigenvalues

Let A be a 2×2 matrix, and let λ be an eigenvalue of A.

Then $A - \lambda I$ is not invertible, so the second row is *automatically* a multiple of the first. (Think about it for a while: otherwise the rref is $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$.)

Hence the second row disappears in the rref, so *we don't care what it is!*

If $A - \lambda I = \begin{pmatrix} a & b \\ \star & \star \end{pmatrix}$, then $(A - \lambda I)\begin{pmatrix} b \\ -a \end{pmatrix} = 0$, so $\begin{pmatrix} b \\ -a \end{pmatrix}$ is an eigenvector. So is $\begin{pmatrix} -b \\ a \end{pmatrix}$.

Example:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \lambda = \frac{1-i}{\sqrt{2}}.$$

Then:

$$A - \lambda I = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -1 \\ \star & \star \end{pmatrix}$$

so an eigenvector is

$$v = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Conjugate Eigenvectors

For $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$,

$$\text{the eigenvalue } \frac{1+i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

$$\text{the eigenvalue } \frac{1-i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Do you notice a pattern?

Fact

Let A be a real square matrix. If λ is an eigenvalue with eigenvector v, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$.

Why?

$$Av = \lambda \implies A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\overline{v}.$$

Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.

A 3×3 Example

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is

$$f(\lambda) = \det \begin{pmatrix} \frac{4}{5} - \lambda & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} - \lambda & 0 \\ 0 & 0 & 2 - \lambda \end{pmatrix} = \boxed{(2 - \lambda)} \left(\lambda^2 - \frac{8}{5}\lambda + 1 \right).$$

This factors out automatically if you expand cofactors along the third row or column

We computed the roots of this polynomial (times 5) before:

$$\lambda = \quad 2, \quad \frac{4 + 3i}{5}, \quad \frac{4 - 3i}{5}.$$

We eyeball an eigenvector with eigenvalue 2 as $(0, 0, 1)$.

A 3×3 Example

Continued

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

To find the other eigenvectors, we row reduce:

$$A - \frac{4+3i}{5}I = \begin{pmatrix} -\frac{3}{5}i & -\frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5}i & 0 \\ 0 & 0 & 2 - \frac{4+3i}{5} \end{pmatrix} \xrightarrow{\text{scale rows}} \begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second row is i times the first:

$$\xrightarrow{\text{row replacement}} \begin{pmatrix} -i & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{divide by } -i\text{,swap}} \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, $z = 0$, so an eigenvector is $\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$. Therefore, an eigenvector with conjugate eigenvalue $\frac{4-3i}{5}$ is $\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

Theorem

Let A be a 2×2 matrix with complex (non-real) eigenvalue λ, and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}.$$

The matrix C is a composition of rotation by $-\arg(\lambda)$ and scaling by $|\lambda|$:

$$C = \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{pmatrix} \begin{pmatrix} \cos(-\arg(\lambda)) & -\sin(-\arg(\lambda)) \\ \sin(-\arg(\lambda)) & \cos(-\arg(\lambda)) \end{pmatrix}.$$

A 2×2 matrix with complex eigenvalue λ is similar to (rotation by the argument of $\overline{\lambda}$) composed with (scaling by $|\lambda|$). This is multiplication by $\overline{\lambda}$ in $\mathbf{C} \sim \mathbf{R}^2$.

Geometric Interpretation of Complex Eigenvalues

2×2 example

What does $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ do geometrically?

- The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \operatorname{Tr}(A)\,\lambda + \det(A) = \lambda^2 - 2\lambda + 2.$$

  The roots are

$$\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

- Let $\lambda = 1 - i$. We compute an eigenvector v:

$$A - \lambda I = \begin{pmatrix} i & -1 \\ \star & \star \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

- Therefore, $A = PCP^{-1}$ where

$$P = \left(\operatorname{Re}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \operatorname{Im}\begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Geometric Interpretation of Complex Eigenvalues

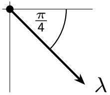
2×2 example, continued

$$A = C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \lambda = 1 - i$$

- The matrix $C = A$ scales by a factor of

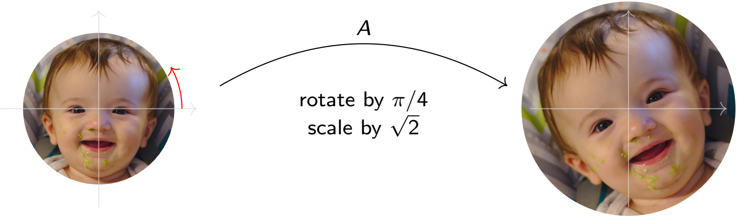
$$|\lambda| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

- The argument of λ is $-\pi/4$:



  Therefore $C = A$ rotates by $+\pi/4$.

- (We already knew this because $A = \sqrt{2}$ times the matrix for rotation by $\pi/4$ from before.)



What does $A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?

- The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - 2\sqrt{3}\,\lambda + 4.$$

  The roots are

$$\frac{2\sqrt{3} \pm \sqrt{12-16}}{2} = \sqrt{3} \pm i.$$

- Let $\lambda = \sqrt{3} - i$. We compute an eigenvector v:

$$A - \lambda I = \begin{pmatrix} 1+i & -2 \\ \star & \star \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}.$$

- It follows that $A = PCP^{-1}$ where

$$P = \left(\mathrm{Re}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \quad \mathrm{Im}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{pmatrix} = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}.$$

Geometric Interpretation of Complex Eigenvalues

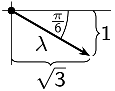
Another 2×2 example, continued

$$A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix} \qquad C = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \qquad \lambda = \sqrt{3} - i$$

- The matrix C scales by a factor of

$$|\lambda| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2.$$

- The argument of λ is $-\pi/6$:

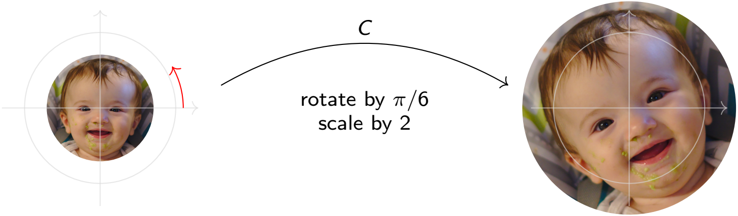


Therefore C rotates by $+\pi/6$.

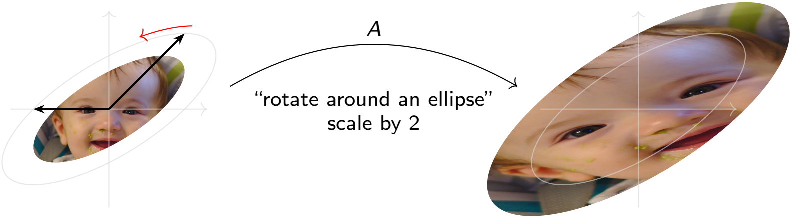
Geometric Interpretation of Complex Eigenvalues

Another 2×2 example: picture

What does $A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?



$A = PCP^{-1}$ does the same thing, but with respect to the basis $\mathcal{B} = \left\{ \binom{1}{1}, \binom{-1}{0} \right\}$ of columns of P:



Classification of 2×2 Matrices with a Complex Eigenvalue

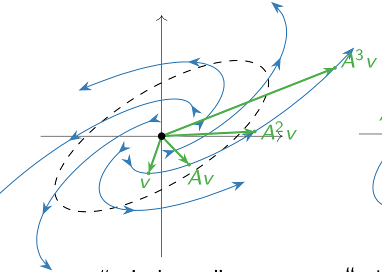
Triptych

Let A be a real matrix with a complex eigenvalue λ. One way to understand the geometry of A is to consider the difference equation $v_{n+1} = Av_n$, i.e. the sequence of vectors $v, Av, A^2v, \ldots$

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$$

$$\lambda = \frac{\sqrt{3}-i}{\sqrt{2}}$$

$|\lambda| > 1$

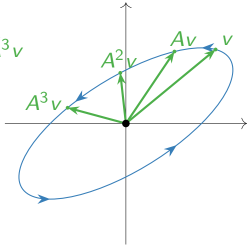


"spirals out"

$$A = \frac{1}{2}\begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$$

$$\lambda = \frac{\sqrt{3}-i}{2}$$

$|\lambda| = 1$

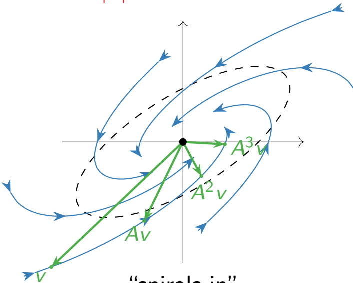


"rotates around an ellipse"

$$A = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$$

$$\lambda = \frac{\sqrt{3}-i}{2\sqrt{2}}$$

$|\lambda| < 1$



"spirals in"

Theorem

Let A be a 2×2 matrix with complex eigenvalue $\lambda = a + bi$ (where $b \neq 0$), and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{pmatrix} \quad \text{and} \quad C = (\text{rotation}) \cdot (\text{scaling}).$$

This is very analogous to diagonalization. In the 2×2 case:

Theorem

Let A be a 2×2 matrix with linearly independent eigenvectors v_1, v_2 and associated eigenvalues λ_1, λ_2. Then

$$A = PDP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

scale x-axis by λ_1
scale y-axis by λ_2

Picture with 2 Real Eigenvalues

We can draw analogous pictures for a matrix with 2 real eigenvalues.

Example: Let $A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$.

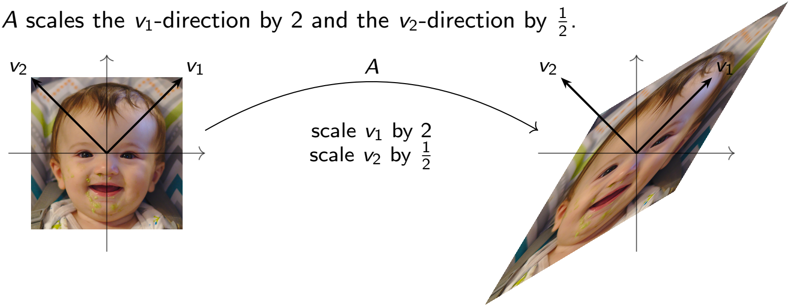
This has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$, with eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore, $A = PDP^{-1}$ with

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

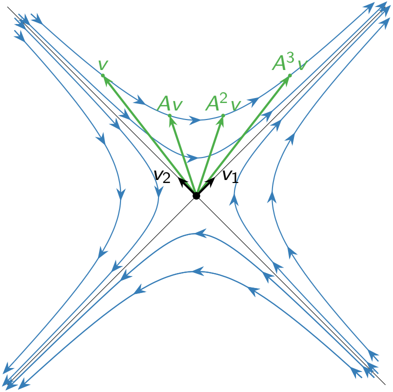
So A scales the v_1-direction by 2 and the v_2-direction by $\frac{1}{2}$.



Picture with 2 Real Eigenvalues

We can also draw a picture from the perspective a difference equation: in other words, we draw $v, Av, A^2v, \ldots$

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad \begin{matrix} \lambda_1 = 2 \\ |\lambda_1| > 1 \end{matrix} \qquad \begin{matrix} \lambda_2 = \frac{1}{2} \\ |\lambda_1| < 1 \end{matrix}$$



Exercise: Draw analogous pictures when $|\lambda_1|, |\lambda_2|$ are any combination of $< 1, = 1, > 1$.

The Higher-Dimensional Case

Theorem

Let A be a real $n \times n$ matrix. Suppose that for each (real or complex) eigenvalue, the dimension of the eigenspace equals the algebraic multiplicity. Then $A = PCP^{-1}$, where P and C are as follows:

1. C is **block diagonal**, where the blocks are 1×1 blocks containing the real eigenvalues (with their multiplicities), or 2×2 blocks containing the matrices $\begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}$ for each non-real eigenvalue λ (with multiplicity).
2. The columns of P form bases for the eigenspaces for the real eigenvectors, or come in pairs $(\operatorname{Re} v \;\; \operatorname{Im} v)$ for the non-real eigenvectors.

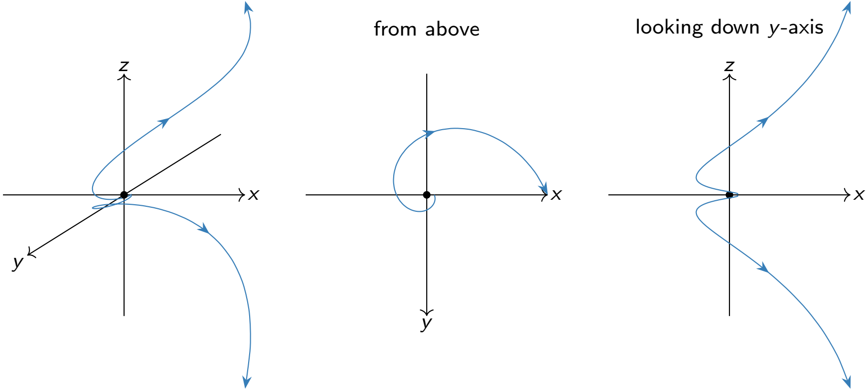
For instance, if A is a 3×3 matrix with one real eigenvalue λ_1 with eigenvector v_1, and one conjugate pair of complex eigenvalues $\lambda_2, \overline{\lambda}_2$ with eigenvectors $v_2, \overline{v}_2$, then

$$P = \begin{pmatrix} | & | & | \\ v_1 & \operatorname{Re} v_2 & \operatorname{Im} v_2 \\ | & | & | \end{pmatrix} \qquad C = \begin{pmatrix} \boxed{\lambda_1} & 0 & 0 \\ 0 & \operatorname{Re}\lambda_2 & \operatorname{Im}\lambda_2 \\ 0 & -\operatorname{Im}\lambda_2 & \operatorname{Re}\lambda_2 \end{pmatrix}$$

The Higher-Dimensional Case

Example

Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. This acts on the xy-plane by rotation by $\pi/4$ and scaling by $\sqrt{2}$. This acts on the z-axis by scaling by 2. Pictures:



Remember, in general $A = PCP^{-1}$ is only *similar* to such a matrix C: so the x, y, z axes have to be replaced by the columns of P.