Math 1553 Worksheet §1.3, interactive supplement

Solutions

If you don't have a computer, find someone who does.

1. Let $v_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} \qquad w = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}$.

Question: Is w a linear combination of v_1 and v_2? In other words, is w in Span$\{v_1, v_2\}$?

a) Formulate this question as a vector equation.

b) Formulate this question as a system of linear equations.

c) Formulate this question as an augmented matrix.

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

Solution.

a) Does the following vector equation have a solution?

$$x\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}$$

b) Does the following linear system have a solution?

$$\begin{aligned} 2x - 2y &= 2 \\ x - 3y &= -4 \\ 3x - y &= 8 \end{aligned}$$

c) As an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & -2 & 2 \\ 1 & -3 & -4 \\ 3 & -1 & 8 \end{array}\right)$$

e) Row reducing yields

$$\left(\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{array}\right)$$

so $x = 7/2$ and $y = 5/2$.

2. Consider the system of linear equations

$$\begin{aligned} x + 2y &= 7 \\ 2x + y &= -2 \\ -x - y &= 4 \end{aligned}$$

Question: Does this system have a solution? If so, what is the solution set?

a) Formulate this question as an augmented matrix.

b) Formulate this question as a vector equation.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

Solution.

a) As an augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 1 & -2 \\ -1 & -1 & 4 \end{array}\right)$$

b) What are the solutions to the following vector equation?

$$x\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$$

c) There exists a solution if and only if $\begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

e) Row reducing yields

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right),$$

so there are no solutions. (This should be obvious from the picture in (d)).

3. Consider the vector equation

$$x\begin{pmatrix}2\\1\\3\end{pmatrix}+y\begin{pmatrix}-2\\-1\\-1\end{pmatrix}+z\begin{pmatrix}3\\0\\4\end{pmatrix}=\begin{pmatrix}-5\\-1\\-2\end{pmatrix}.$$

Question: Is there a solution? If so, what is the solution set?

a) Formulate this question as an augmented matrix.

b) Formulate this question as a system of linear equations.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

Solution.

a) As an augmented matrix:

$$\left(\begin{array}{ccc|c}2 & -2 & 3 & -5\\1 & -1 & 0 & -1\\3 & -1 & 4 & -2\end{array}\right)$$

b) What is the solution set of the following linear system?

$$\begin{aligned}2x - 2y + 3z &= -5\\ x - y &= -1\\ 3x - y + 4z &= -2\end{aligned}$$

c) There exists a solution if and only if $\begin{pmatrix}-5\\-1\\-2\end{pmatrix}$ is in Span $\left\{\begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}-2\\-1\\-1\end{pmatrix}, \begin{pmatrix}3\\0\\4\end{pmatrix}\right\}$.

e) Row reducing yields

$$\left(\begin{array}{ccc|c}1 & 0 & 0 & 3/2\\0 & 1 & 0 & 5/2\\0 & 0 & 1 & -1\end{array}\right),$$

so $x = 3/2$, $y = 5/2$, and $z = -1$.

4. Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 1 & -3 & -4 & -9 \\ 3 & -1 & 8 & 9 \end{array}\right)$$

Question: Does the corresponding linear system have a solution? If so, what is the solution set?

a) Formulate this question as a vector equation.

b) Formulate this question as a system of linear equations.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

f) Find a **different** solution in parts (e) and (d).

Solution.

a) What are the solutions to the following vector equation?

$$x\begin{pmatrix}2\\1\\3\end{pmatrix}+y\begin{pmatrix}-2\\-3\\-1\end{pmatrix}+z\begin{pmatrix}2\\-4\\8\end{pmatrix}=\begin{pmatrix}0\\-9\\9\end{pmatrix}$$

b) What is the solution set of the following linear system?

$$\begin{aligned} 2x - 2y + 2z &= 0 \\ x - 3y - 4z &= -9 \\ 3x - y + 8z &= 9 \end{aligned}$$

c) There exists a solution if and only if $\begin{pmatrix}0\\-9\\9\end{pmatrix}$ is in Span $\left\{\begin{pmatrix}2\\1\\3\end{pmatrix},\begin{pmatrix}-2\\-3\\-1\end{pmatrix},\begin{pmatrix}2\\-4\\8\end{pmatrix}\right\}$.

e) Row reducing yields

$$\left(\begin{array}{ccc|c} 1 & 0 & 7/2 & 9/2 \\ 0 & 1 & 5/2 & 9/2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Hence z is a free variable, so the solution in parametric form is

$$\begin{aligned} x &= \tfrac{9}{2} - \tfrac{7}{2}z \\ y &= \tfrac{9}{2} - \tfrac{5}{2}z. \end{aligned}$$

Taking $z = 0$ yields the solution $x = y = 9/2$.

f) Taking $z = 1$ yields the solution $x = 1, y = 2$.

5. Decide if each of the following statements is true or false. If it is true, prove it; if it is false, provide a counterexample.

a) Every set of four or more vectors in $\mathbf{R}^3$ will span $\mathbf{R}^3$.

b) The span of any set contains the zero vector.

Solution.

a) This is **false.** For instance, the vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \right\}$$

only span the x-axis.

b) This is **true.** We have

$$0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_p.$$

Aside: the span of the empty set is equal to $\{0\}$, because 0 is the empty sum, i.e. the sum with no summands. Indeed, if you add the empty sum to a vector v, you get $v +$ (no other summands), which is just v; and the only vector which gives you v when you add it to v, is 0. (If you find this argument intriguing, you might want to consider taking abstract math courses later on.)