Math 1553 Worksheet §5.3, 5.5

1. Answer yes / no / maybe. In each case, A is a matrix whose entries are real.

a) If A is a 3×3 matrix with characteristic polynomial $-\lambda(\lambda-5)^2$, then the 5-eigenspace is 2-dimensional.

b) If A is an invertible 2×2 matrix, then A is diagonalizable.

c) Can a 3×3 matrix A have a non-real complex eigenvalue with multiplicity 2?

d) Can a 3×3 matrix A have eigenvalues $3, 5$, and $2+i$?

Solution.

a) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5- eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(\lambda-5)^2$.

b) Maybe. The identity matrix is invertible and diagonalizable, but the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

c) No. If c is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\bar{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean A has a characteristic polynomial of degree 4 or more, which is impossible for a 3×3 matrix.

d) No. If $2+i$ is an eigenvalue then so is its conjugate $2-i$.

2. Let $A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix}$.

The characteristic polynomial for A is $-\lambda^3 + 7\lambda^2 - 16\lambda + 12$, and $\lambda - 3$ is a factor. Decide if A is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution.

By polynomial division,

$$\frac{-\lambda^3 + 7\lambda^2 - 16\lambda + 12}{\lambda - 3} = -\lambda^2 + 4\lambda - 4 = -(\lambda-2)^2.$$

Thus, the characteristic poly factors as $-(\lambda-3)(\lambda-2)^2$, so the eigenalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

For $\lambda_1 = 3$, we row-reduce $A-3I$:

$$\begin{pmatrix} 5 & 36 & 62 \\ -6 & -37 & -62 \\ 3 & 18 & 30 \end{pmatrix} \xrightarrow[(\text{New } R_1)/3]{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 6 & 10 \\ -6 & -37 & -62 \\ 5 & 36 & 62 \end{pmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & 6 & 10 \\ 0 & -1 & -2 \\ 0 & 6 & 12 \end{pmatrix}$$

$$\xrightarrow[\text{then } R_2 = -R_2]{R_3 = R_3 + 6R_2} \begin{pmatrix} 1 & 6 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 - 6R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the solutions to $\left(A-3I \mid 0\right)$ are $x_1 = 2x_3,\ \ x_2 = -2x_3,\ \ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}. \qquad \text{The 3-eigenspace has basis } \left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda_2 = 2$, we row-reduce $A-2I$:

$$\begin{pmatrix} 6 & 36 & 62 \\ -6 & -36 & -62 \\ 3 & 18 & 31 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & 6 & \frac{31}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solutions to $\left(A-2I \quad 0\right)$ are $x_1 = -6x_2 - \frac{31}{3}x_3,\ x_2 = x_2,\ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6x_2 - \frac{31}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix}.$$

The 2-eigenspace has basis $\left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$.

Therefore, $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 2 & -6 & -\frac{31}{3} \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that we arranged the eigenvectors in P in order of the eigenvalues $3,\ 2,\ 2$, so we had to put the diagonals of D in the same order.

3. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

a) Find all eigenvalues and eigenvectors of A.

b) Write $A = PCP^{-1}$, where C is a rotation followed by a scale. Describe what A does geometrically. Draw a picture.

Solution.

a) The characteristic polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

For the eigenvalue $\lambda = 1 - 2i$, we row-reduce $\left(A - (1-2i)I \mid 0\right)$.

$$\left(\begin{array}{cc|c} 2i & 2 & 0 \\ -2 & 2i & 0 \end{array}\right) \xrightarrow{R_1 = R_1 \cdot 1/2i} \left(\begin{array}{cc|c} 1 & -i & 0 \\ -2 & 2i & 0 \end{array}\right) \xrightarrow{R_2 = R_2 + 2R_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right).$$

So $x_1 = ix_2$ and $x_2 = x_2$. A corresponding eigenvector is $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$, and any nonzero complex multiple of v will also be an eigenvector.
(If we used the 2×2 trick from the 5.5 slides, we would have found that an eigenvector is $\begin{pmatrix} 2 \\ -2i \end{pmatrix}$, which is really just $-2i$ times the eigenvector v above.)

From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue $\lambda = 1 + 2i$, a corresponding eigenvector is $w = \overline{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

b) We use $\lambda = 1 - 2i$ and its associated $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

$A = PCP^{-1}$ where $P = \left(\mathrm{Re}(v) \quad \mathrm{Im}(v)\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$C = \begin{pmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

The scale is by a factor of $|\lambda| = |1 + 2i| = \sqrt{1^2 + 2^2} = \sqrt{5}$. If we factor this out of C we get

$$C = \sqrt{5} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}.$$

We see $\cos(\theta) = \frac{1}{\sqrt{5}}$ and $\sin(\theta) = \frac{2}{\sqrt{5}}$, so $\tan(\theta) = 2$ and $\theta = \arctan(2)$.

C is rotation by the angle $\arctan(2)$, followed by scaling by a factor of $\sqrt{5}$.

See the [interactive] demo for how A acts geometrically.

***Note: there are multiple answers possible for part **b)**.
For example, the 2×2 trick from the 5.5 slides says that if λ is an eigenvalue of A, then one eigenvector is $\begin{pmatrix} b \\ -a \end{pmatrix}$ where $\begin{pmatrix} a & b \end{pmatrix}$ is the first row of $A-\lambda I$.

Row 1 of $A-\lambda I$ was $\begin{pmatrix} 2i & 2 \end{pmatrix}$, so $\begin{pmatrix} 2 \\ -2i \end{pmatrix}$ as an eigenvector.

This would give us $P=\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ rather than $P=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. However, it would still be the case that $A=PCP^{-1}$ since

$$PCP^{-1}=\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}=A.$$

Supplemental Problems

For those who want additional practice problems after completing the worksheet, here are some extra practice problems.

1. Let A and B be 3×3 real matrices. Answer yes / no / maybe:

a) If A and B have the same eigenvalues, then A is similar to B.

b) If A and B both have eigenvalues $-1, 0, 1$, then A is similar to B.

c) If A is diagonalizable and invertible, then A^{-1} is diagonalizable.

Solution.

a) Maybe. For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ have the same eigenvalues ($\lambda = 0$ with alg. multiplicity 2) but are not similar, whereas $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is similar to itself.

b) Yes. In this case, A and B are 3×3 matrices with 3 distinct eigenvalues and thus automatically diagonalizable, and each is similar to $D=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Since A and D are similar, and B and D are similar, it follows that A and B are similar.

$$A=PDP^{-1} \quad B=QDQ^{-1} \qquad A=PDP^{-1}=PQ^{-1}BQP^{-1}=PQ^{-1}B(PQ^{-1})^{-1}.$$

c) Yes. If $A=PDP^{-1}$ and A is invertible then its eigenvalues are all nonzero, so the diagonal entries of D are nonzero and thus D is invertible (pivot in every diagonal position). Thus, $A^{-1}=(PDP^{-1})^{-1}=(P^{-1})^{-1}D^{-1}P^{-1}=PD^{-1}P^{-1}$.

2. Give an example of a non-diagonal 2×2 matrix which is diagonalizable but not invertible. Justify your answer.

Solution.

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not invertible (row of zeros) but is diagonalizable since its has two distinct eigenvalues 0 and 1 (it is triangular, so its diagonals are its eigenvalues)

3. Suppose A is a 7×7 matrix with four distinct eigenvalues. One eigenspace has dimension 2, while another eigenspace has dimension 3. Is it possible that A is not diagonalizable?

Solution.

A must be diagonalizable. It is a general fact that every eigenvalue of a matrix has a corresponding eigenspace which is at least 1-dimensional. Given this and the fact that A has four total eigenvalues, we see the sum of dimensions of the eigenspaces of A is at least $2 + 3 + 1 + 1 = 7$, and in fact must equal 7 since that is the max possible for a 7×7 matrix. Therefore, A has 7 linearly independent eigenvectors and is therefore diagonalizable.

4. Let $A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$.

a) Find all (complex) eigenvalues and eigenvectors of A.

b) Write $A = PCP^{-1}$, where C is a block diagonal matrix, as in the slides near the end of section 5.5.

c) What does A do geometrically? Draw a picture.

Solution.

a) First we compute the characteristic polynomial by expanding cofactors along the third row:

$$f(\lambda) = \det\begin{pmatrix} 4-\lambda & -3 & 3 \\ 3 & 4-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{pmatrix}$$
$$= (2-\lambda)\big((4-\lambda)^2 + 9\big) = (2-\lambda)(\lambda^2 - 8\lambda + 25).$$

Using the quadratic equation on the second factor, we find the eigenvalues

$$\lambda_1 = 2 \qquad \lambda_2 = 4 - 3i \qquad \overline{\lambda}_2 = 4 + 3i.$$

Next compute an eigenvector with eigenvalue $\lambda_1 = 2$:

$$A - 2I = \begin{pmatrix} 2 & -3 & 3 \\ 3 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x = 0$, $y = z$, so the parametric vector form of the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{eigenvector}} v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Now we compute an eigenvector with eigenvalue $\lambda_2 = 4 - 3i$:

$$A = (4-3i)I = \begin{pmatrix} 3i & -3 & 3 \\ 3 & 3i & -2 \\ 0 & 0 & 3i-2 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 3 & 3i & -2 \\ 3i & -3 & 3 \\ 0 & 0 & 3i-2 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - iR_1} \begin{pmatrix} 3 & 3i & -2 \\ 0 & 0 & 3+2i \\ 0 & 0 & 3i-2 \end{pmatrix} \xrightarrow{R_2 = R_2 \div (3+2i)} \begin{pmatrix} 3 & 3i & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 3i-2 \end{pmatrix}$$

$$\xrightarrow{\text{row replacements}} \begin{pmatrix} 3 & 3i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 \div 3} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form of the solution is $x = -iy$, $z = 0$, so the parametric vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{eigenvector}} v_2 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.$$

An eigenvector for the complex conjugate eigenvalue $\overline{\lambda}_2 = 4+3i$ is the complex conjugate eigenvector $\overline{v}_2 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.

b) According to the "block-diagonalization" theorem, we have $A = PCP^{-1}$ where

$$P = \begin{pmatrix} | & | & | \\ \operatorname{Re} v_2 & \operatorname{Im} v_2 & v_1 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$C = \begin{pmatrix} \operatorname{Re} \lambda_2 & \operatorname{Im} \lambda_2 & 0 \\ -\operatorname{Im} \lambda_2 & \operatorname{Re} \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(I've ordered the eigenvalues in this way to make the picture look nicer in my "z is up" coordinate system.)

c) The matrix C scales by 2 in the z-direction, and rotates by $\arg(-\lambda_2) = \arctan(3/4) \sim .6435$ radians and scales by $|\lambda_2| = \sqrt{4^3 + 3^3} = 5$ in the xy-directions. The matrix A does the same thing, with respect to the basis

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

of columns of P. [interactive]