MATH 1553
PRACTICE MIDTERM 1 (VERSION B)

Name		**Section**	

1	2	3	4	5	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

Problem 1.

[2 points each]

In this problem, A is an $m \times n$ matrix (m rows and n columns) and b is a vector in $\mathbf{R}^m$. Circle **T** if the statement is always true (for any choices of A and b) and circle **F** otherwise. Do not assume anything else about A or b except what is stated.

a) **T** **F** The matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is in reduced row echelon form.

b) **T** **F** If A has fewer than m pivots then $Ax = b$ has infinitely many solutions.

c) **T** **F** If b is in the span of the columns of A, then $Ax = b$ is consistent.

d) **T** **F** The zero vector is in the span of the columns of A.

e) **T** **F** If $Ax = b$ is consistent, then b is in the span of the columns of A.

Solution.

a) True.

b) False: for instance,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

has a unique solution.

c) True: $Ax = b$ is consistent if and only if b is in the span of the columns of A.

d) True: the zero vector is in any span.

e) True: $Ax = b$ is consistent if and only if b is in the span of the columns of A.

Problem 2.

Consider the matrix equation $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

a) Find the reduced row echelon form of the augmented matrix $(A \mid b)$.

b) Write the solution set to $Ax = b$ in vector parametric form.

c) Write the solution set to $Ax = b$ as a translate of a span.

d) What best describes the geometric relationship between the solutions to $Ax = 0$ and the solutions to $Ax = b$? (Same A and b as above.)
(1) They are both lines through the origin.
(2) They are parallel lines.
(3) They are both planes through the origin.
(4) They are parallel planes.

Solution.

a)

$$\left(\begin{array}{cccc|c} 1 & 3 & 8 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 4 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 3 & 8 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

b)

$$\begin{aligned} x_1 \quad\quad + 2x_3 \quad\quad &= -2 \\ x_2 + 2x_3 \quad\quad &= 1 \\ x_4 &= 0 \end{aligned} \rightsquigarrow \begin{aligned} x_1 &= -2x_3 - 2 \\ x_2 &= -2x_3 + 1 \\ x_3 &= x_3 \\ x_4 &= 0 \end{aligned}$$

$$\rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

c)

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \operatorname{Span}\left\{ \begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

d) The solution set to $Ax = b$ is a line which does not go through the origin: it is a vector plus the span of a nonzero vector. Hence the solution set to $Ax = 0$ is the parallel line through the origin, since the solution set to $Ax = b$ is obtained from the solution set to $Ax = 0$ by translating.

Problem 3.

[10 points]

Find all values of k such that the following vector equation has a unique solution:

$$x\begin{pmatrix}-1\\3\\-1\end{pmatrix}+y\begin{pmatrix}1\\1\\-1\end{pmatrix}+z\begin{pmatrix}1\\k\\-7\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Solution.

We form an augmented matrix and row reduce:

$$\left(\begin{array}{ccc|c}-1&1&1&0\\3&1&k&0\\-1&-1&-7&0\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|c}-1&1&1&0\\0&4&k+3&0\\0&-2&-8&0\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|c}-1&1&1&0\\0&-2&-8&0\\0&4&k+3&0\end{array}\right)$$

$$\rightsquigarrow\left(\begin{array}{ccc|c}-1&1&1&0\\0&-2&-8&0\\0&0&k-13&0\end{array}\right)$$

This matrix is in row echelon form. It has pivots in the first two columns, and it has a pivot in the third if and only if $k \neq 13$. Hence are no free variables if $k \neq 13$, in which case there is a unique solution.

Problem 4. [10 points]

Find all values of h such that $\begin{pmatrix}1\\h\\5\end{pmatrix}$ is *not* in the span of $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\4\\1\end{pmatrix}$.

Solution.

The vector $\begin{pmatrix}1\\h\\5\end{pmatrix}$ is in the span of $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\4\\1\end{pmatrix}$ if and only if the vector equation

$$x\begin{pmatrix}1\\3\\2\end{pmatrix}+y\begin{pmatrix}-1\\4\\1\end{pmatrix}=\begin{pmatrix}1\\h\\5\end{pmatrix}$$

has a solution. We put it into an augmented matrix and row reduce:

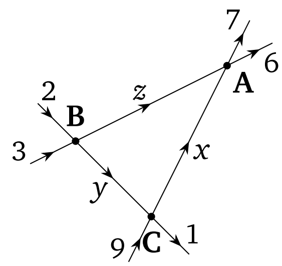
$$\left(\begin{array}{rr|r}1&-1&1\\3&4&h\\2&1&5\end{array}\right)\rightsquigarrow\left(\begin{array}{rr|r}1&-1&1\\0&7&h-3\\0&3&3\end{array}\right)\rightsquigarrow\left(\begin{array}{rr|r}1&-1&1\\0&3&3\\0&7&h-3\end{array}\right)$$

$$\rightsquigarrow\left(\begin{array}{rr|r}1&-1&1\\0&1&1\\0&7&h-3\end{array}\right)\rightsquigarrow\left(\begin{array}{rr|r}1&-1&1\\0&1&1\\0&0&h-10\end{array}\right).$$

This is consistent if and only if $h=10$, so $\begin{pmatrix}1\\h\\5\end{pmatrix}$ is *not* in the span if and only if $h\neq 10$.

Problem 5.

The following diagram indicates traffic flow in one part of town (the numbers indicate the number of cars per minute on each section of road, and the letters indicate intersections):



a) Write a system of linear equations in x, y, and z describing the traffic flow around the triangle.

b) Write the above system of linear equations as an augmented matrix.

c) Write the above system of linear equations as a vector equation.

d) Write the above system of linear equations as a matrix equation.

Solution.

a) We have to have the same amount of traffic going into each intersection as going out. For the intersections **A**, **B**, and **C**, respectively, this means:

$$\begin{aligned} x \quad\;\; + z &= 7+6 = 13 \\ y + z &= 2+3 = \;\;5 \\ x - y \quad\;\; &= 9-1 = \;\;8 \end{aligned}$$

b)

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 13 \\ 0 & 1 & 1 & 5 \\ 1 & -1 & 0 & 8 \end{array}\right)$$

c)

$$x\begin{pmatrix}1\\0\\1\end{pmatrix} + y\begin{pmatrix}0\\1\\-1\end{pmatrix} + z\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}13\\5\\8\end{pmatrix}$$

d)

$$\begin{pmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & -1 & 0\end{pmatrix} x = \begin{pmatrix}13\\5\\8\end{pmatrix}$$

[Scratch work]